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# Critical exponent of a simple model of spot replication

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**Abstract**

This paper is concerned with a semilinear elliptic inhomogeneous equation

$$\Delta u - u + \left(1 + a|x|^q\right) u^p = 0$$

introduced in Chen and Kolokolnikov (2012) [2] as a simple prototype of self-replication in more complex reaction–diffusion systems. Under certain conditions on $p$, $q$, it was previously shown by Chen–Kolokolnikov that the equation has no radial ground state solution when the control parameter $a$ is increased above some threshold. This property is important for the existence of a saddle-node bifurcation proposed in the Nishiura–Ueyema conditions, which is believed to be necessary for an initiation of a self-replication event. In this paper, we generalize Chen–Kolokolnikov's result to non-radial positive solutions by proving a Liouville-type nonexistence theorem. Furthermore we derive a local version of this nonexistence theorem for solutions defined on a bounded ball. Our result indicates that critical values of $q$ derived in Ding and Ni (1986) [3] are also crucial for the existence and nonexistence problem of positive solutions when the space dimension $N \geq 3$.


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*Keywords:* Reaction–diffusion equation; Spot self-replication; Liouville type theorem; Critical exponent

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## 1. Introduction

In this paper we establish a Liouville type theorem for the non-autonomous PDE

$$\Delta u - u + \left(1 + a|x|^q\right) u^p = 0, \quad u > 0, \quad x \in \mathbb{R}^N. \tag{1.1}$$

This PDE was introduced in [2] as a simple prototype example of spike self-replication that is commonplace in many complex reaction–diffusion systems. These include Gray–Scott model [24,23,21,22,20,5,4,16,6], and the related Schnakenberg model [15,25], the Gierer–Meinhardt model [18,7,17], the Belousov–Zhabotinsky reaction [13,19], the ferrocyanide–iodide–sulfite system [12], the Bonhoffer van-der-Pol-type system [10,11] and the Brusselator [14].

The inhomogeneity $a|x|^q$ is intimately related to self-replication phenomenon and roughly speaking, models the effect of the "slow" (inhibitor) component of the two-component reaction–diffusion systems with $u$ modeling the "fast" (activator) component which exhibits spike solutions.

In an effort to classify reaction–diffusion systems that can exhibit pulse self-replication, Nishiura and Ueyema in [21] proposed a set of necessary conditions for this phenomenon to occur. Roughly speaking, their conditions can be stated as follows

- (**S1**) The disappearance of the ground-state solution due to a fold point (saddle-node bifurcation) that occurs when a control parameter is increased above a certain threshold value.
- (**S2**) The existence of a dimple eigenfunction at the fold point, which is believed to be responsible for the initiation of the self-replication process. By definition, a dimple eigenfunction is a radially symmetric eigenfunction $\Phi(|x|)$ associated with a zero eigenvalue at the fold point, that decays as $|x| \to \infty$ and that has a positive zero.
- (**S3**) Stability of the steady-state solution on one side of the fold point.
- (**S4**) The alignment of the fold points, so that the disappearance of $K$ ground states, with $K = 1, 2, 3, \ldots$, occurs at roughly the same value of the control parameter.

These conditions are believed to be necessary (although not sufficient) for an initiation of the self-replication event. They were first verified numerically for a certain regime of the Gray–Scott model in [21,8]. In a different regime, the Gray–Scott model reduces to the so-called core problem [20,6,16]. For this core problem, the existence of a fold point (condition (S1)) in one dimension was demonstrated numerically in [20], and conditions (S2), (S3) were also numerically verified in [16]. More recently, the following weaker version of Condition (S1) was shown analytically in [6]:

- (**S1**$^*$) The steady-state ceases to exist if a control parameter is increased above a certain threshold value.

In [2] two of the authors of this paper considered $a$ in (1.1) as the control parameter. They showed analytically and numerically that the simple model (1.1) can exhibit self-replication for some values of $p$ and $q$ in any dimension as $a$ is sufficiently increased from zero, due to the disappearance of the solution at the fold-point. Also conditions (S1$^*$), (S2) and (S3) were analytically verified.

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