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Devaney chaos, Li–Yorke chaos, and multi-dimensional Li–Yorke chaos for topological dynamics

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Abstract

Let $\pi : T \times X \to X$, written $T \curvearrowright_\pi X$, be a topological semiflow/flow on a uniform space X with T a multiplicative topological semigroup/group not necessarily discrete. We then prove:

- If $T \curvearrowright_\pi X$ is non-minimal topologically transitive with dense almost periodic points, then it is sensitive to initial conditions. As a result of this, Devaney chaos $\Rightarrow$ Sensitivity to initial conditions, for this very general setting.

Let $\mathbb{R}_+ \curvearrowright_\pi X$ be a C^0-semiflow on a Polish space; then we show:

- If $\mathbb{R}_+ \curvearrowright_\pi X$ is topologically transitive with at least one periodic point p and there is a dense orbit with no nonempty interior, then it is *multi-dimensional Li–Yorke chaotic*; that is, there is a uncountable set $\Theta \subseteq X$ such that for any $k \geq 2$ and any distinct points $x_1, \dots, x_k \in \Theta$, one can find two time sequences $s_n \to \infty, t_n \to \infty$ with

$$s_n(x_1, \dots, x_k) \to (x_1, \dots, x_k) \in X^k \quad \text{and} \quad t_n(x_1, \dots, x_k) \to (p, \dots, p) \in \Delta_{X^k}.$$

  Consequently, Devaney chaos $\Rightarrow$ Multi-dimensional Li–Yorke chaos.

Moreover, let X be a non-singleton Polish space; then we prove:

- Any weakly-mixing C^0-semiflow $\mathbb{R}_+ \curvearrowright_\pi X$ is densely multi-dimensional Li–Yorke chaotic.

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- Any minimal weakly-mixing topological flow $T \curvearrowright_\pi X$ with T abelian is densely multi-dimensional Li–Yorke chaotic.
- Any weakly-mixing topological flow $T \curvearrowright_\pi X$ is densely Li–Yorke chaotic.

We in addition construct a completely Li–Yorke chaotic minimal $\mathrm{SL}(2,\mathbb{R})$-acting flow on the compact metric space $\mathbb{R} \cup \{\infty\}$. Our various chaotic dynamics are sensitive to the choices of the topology of the phase semigroup/group T.

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1. Introduction

We begin by briefly recalling the definition of topological dynamical systems. Let T be an arbitrary multiplicative topological semigroup not necessarily discrete, which jointly continuously acts from left on a topological space X via an acting map $\pi\colon (t,x) \mapsto \pi(t,x) = \pi_t(x)$, written as

$$\pi\colon T \times X \to X \quad \text{or} \quad T \curvearrowright_\pi X,$$

such that

- $\pi_s\pi_t(x) = \pi_{st}(x)\ \forall s,t \in T, x \in X$ and
- $\pi_e = i_X\colon x \mapsto x$ is the identity map of X onto itself if e is the neutral element of T whenever T is a monoid.

For simplicity, sometimes we will identify the transition map $\pi_t\colon x \mapsto \pi(t,x)$ with $t\colon x \mapsto tx$, for each $t \in T$, if no confusion arises.

Here the triple (T, X, π), or simply (T, X) and $T \curvearrowright_\pi X$, will be called a *topological semiflow* with phase space X and with phase semigroup T. If T is just a topological group, then (T, X, π) or $T \curvearrowright_\pi X$ will be called a *topological flow* with phase group T.

There are systematical studies on chaos for $\mathbb{Z}$- or $\mathbb{Z}_+$-action dynamical systems on compact metric spaces; see, e.g., [28,2,33] and references therein. In this paper, we will study Devaney chaos, Li–Yorke chaos, and multi-dimensional Li–Yorke chaos of flows or semiflows (T, X, π) on uniform spaces X with general topological phase group or semigroup T.

First of all, it should be noticed that chaotic dynamics are very different for general group or semigroup actions, even for $\mathbb{R}$ and $\mathbb{R}_+$, with the well-known $\mathbb{Z}$- or $\mathbb{Z}_+$-actions. Let's see an example, which is already beyond the setting of the following important chaos criterion:

- Let $f\colon X \to X$ be a topologically transitive continuous self-map of a compact metric space X with no isolated points. If there is some subsystem (Y, f) of (X, f) such that $(X \times Y, f)$ is topologically transitive, then (X, f) is densely uniformly (Li–Yorke) chaotic. See [2, Theorem 3.1].