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The cyclicity of period annuli for a class of cubic Hamiltonian systems with nilpotent singular points ☆

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Abstract

This paper deals with the limit cycles of a class of cubic Hamiltonian systems under polynomial perturbations. We suppose that the corresponding Hamiltonian system which has at least one center has finite singular points and is symmetrical with respect to both x-axis and y-axis, and also the origin is a nilpotent singular point. Hence, the Hamiltonian $H(x, y)$ can be written as

$$H(x, y) = -x^2 + (ax^4 + bx^2y^2 + cy^4),\ \ (a, b, c) \in \mathbb{R}^3, c \neq 0.$$

For the above $H(x, y)$, the corresponding vector field is

$$\dot{x} = 2y(bx^2 + 2cy^2),\ \ \dot{y} = 2x(1 - 2ax^2 - by^2).$$

We first obtain that the above vector fields with at least one center can be divided into 8 classes by its topological phase portraits. For the following perturbed system

$$\dot{x} = 2y(bx^2 + 2cy^2) + \varepsilon f(x, y),\ \ \dot{y} = 2x(1 - 2ax^2 - by^2) + \varepsilon g(x, y),$$

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where $0 < |\varepsilon| \ll 1$, $f(x, y)$ and $g(x, y)$ are polynomials in (x, y) with degree n, we give an estimation of the number of isolated zeros of the corresponding Abelian integral. The number of limit cycles follows from Poincaré–Pontryagin Theorem.

Keywords: Hamiltonian system; Abelian integral; Weakened Hilbert's 16th problem; Picard–Fuchs equation; Chebyshev space

1. Introduction and the main results

Probably three of the main problems in the qualitative theory of real planar polynomial differential systems are the determination of limit cycles, the center-focus problem and the first integrals. A limit cycle is an isolated closed orbit of the planar differential system. This paper deals with the limit cycles of a class of symmetric cubic Hamiltonian systems under polynomial perturbations.

Consider the following system:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} + \varepsilon f(x, y), \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x} + \varepsilon g(x, y) \tag{1.1}$$

where $H(x, y)$ is a polynomial of degree $m + 1$, $f(x, y)$ and $g(x, y)$ are polynomials of degree n, and $0 < |\varepsilon| \ll 1$. The weakened Hilbert's 16th problem (see [1]) asks the maximal number of zeros, denoted by $Z(m, n)$, of the following Abelian integral

$$I(h) = \oint_{\Gamma_h} g(x, y)dx - f(x, y)dy, \quad h \in \Sigma, \tag{1.2}$$

where Γ_h is a family of closed orbits defined by the Hamiltonian $H(x, y) = h$, $h \in \Sigma$, and Σ is the maximal open interval (or an union of several maximal open intervals) on which Γ_h is defined. By Poincaré–Pontryagin Theorem, the number of zeros of $I(h)$ provides an upper bound for the number of limit cycles of system (1.1) if $I(h)$ is not identically zero.

The general result of finding $Z(m, n)$ was achieved by Khovansky [16] and Varchenko [24], who proved independently the finiteness of $Z(m, n)$, but no explicit expression of $Z(m, n)$ has been obtained. For the case of $m = 2$, it is known that if the unperturbed system has annulus, then the corresponding Hamiltonian $H(x, y)$ can be written as the following normal form

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3.$$

Horozov and Iliev [14] proved $Z(2, n) \le 5n + 15$ by analyzing the corresponding Picard–Fuchs equations.

For the case of $m \ge 3$, there are some results for the specific Hamiltonians $H(x, y)$. For example, Petrov [22] estimated the number of zeros of $I(h)$ for Hamiltonians

$$H(x, y) = y^2 + x^2 - x^4, \quad \text{and} \quad H(x, y) = y^2 - 2x^2 + x^4.$$