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Dependence of eigenvalues on the boundary conditions of Sturm–Liouville problems with one singular endpoint

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Abstract

This paper is concerned with continuous and differentiable dependence of isolated eigenvalues on the boundary conditions of self-adjoint Sturm–Liouville problems with one singular endpoint. Locally continuous dependence of eigenvalues on the boundary conditions is proved. Especially, in the limit point case, the continuous dependence of isolated eigenvalues is shown by the relationships between Weyl–Titchmarsh $m(\lambda)$-function and the spectrum of the singular problems. Then continuous eigenvalue branches through each isolated eigenvalue over the space of boundary conditions are formed. It is rigorously shown that the eigenfunctions for the eigenvalues along a continuous simple eigenvalue branch can be continuously chosen with uniform bound in L_w^2 norm. Its proof is different from that in the regular Sturm–Liouville problems. The derivative formulas of the continuous simple eigenvalue branch with respect to all the parameters in the boundary conditions are given, and thus its monotonicity with respect to some parameters is derived.

MSC: 34B09; 34B24; 34L15

Keywords: Continuous eigenvalue branch; Singular Sturm–Liouville problem; Boundary condition; Limit point case; Limit circle case

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1. Introduction

A self-adjoint Sturm–Liouville problem (briefly, SLP) considered in the present paper consists of a differential equation

$$-(py')' + qy = \lambda wy, \qquad \text{on } I = (a, b), \tag{1.1}$$

and a self-adjoint boundary condition, where $-\infty \le a < b \le \infty$;

$$p(t) > 0, w(t) > 0 \text{ a.e. on } I, \;\; 1/p, q \in L_{loc}(I, \mathbb{R}); \tag{1.2}$$

and λ is the spectral parameter. Throughout this paper, we assume that a is regular and b is singular (see Definition 2.1). The case that a is singular and b is regular can be discussed similarly.

A weighted Sturm–Liouville problem lies at the basis of the integrability of a celebrated recent model for shallow water waves, which is the following Camassa–Holm (briefly, CH) equation:

$$u_t - u_{xxt} + 3uu_x + 2\kappa u_x = 2u_x u_{xx} + uu_{xxx},$$

where u is the fluid velocity in the x-direction and κ is a constant related to dispersion (see [2,3,8–10] and their references). Fuchssteiner and Fokas first found this equation and derived it as a bi-Hamiltonian system [11]. Later, Cammasa and Holm rediscovered it and served as an integrable model for shallow water waves [6]. To study the integrability of the CH equation, a key point is to understand the following associated spectral problem:

$$-y'' + \frac{1}{4}y = \lambda wy$$

with a certain boundary condition, where $w = u - u_{xx} + \kappa$. The purpose of this paper is to investigate how isolated eigenvalues of a class of general weighted self-adjoint Sturm–Liouville problem depend on the boundary conditions.

Perturbation theory of eigenvalues of regular self-adjoint Sturm–Liouville problems has been extensively investigated (cf. [1,14,16,17,19–22,24,27–29,32]). Kong, Wu, and Zettl showed the continuous dependence of the eigenvalues on the boundary conditions, formed the continuous eigenvalue branches, and gave the derivative formulas of the continuous eigenvalue branches [20,22]. They also completely characterized the discontinuity of the n-th eigenvalue on the boundary conditions [19]. For the general regular Sturm–Liouville problems on time scales, Kong showed that the n-th eigenvalue depends continuously on the separated boundary conditions except at some special ones [17].

However, the related perturbation results for the singular problem are not so rich. If the singular endpoint is in the limit circle and non-oscillatory case, then it can be transformed into a regular one [25] and the corresponding two problems have the same eigenvalues. So many perturbation results in this case can be obtained directly by the corresponding regular problem [1,18,31,32]. If the singular endpoint is in the limit circle and oscillatory case or in the limit point case, there are no desired perturbation results, but some heuristic ones. Brown et al. showed the existence of eigenvalues below the essential spectrum in [4,5] if one singular endpoint is in the limit point case. Zhang et al. showed the n-th eigenvalue below the lower bound of the essential spectrum is continuously dependent on all the parameters in the boundary conditions and gave its derivative formulas [31].

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