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# Translation-invariant estimates for operators with simple characteristics

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**Abstract**

We prove $L^2$ estimates and solvability for a variety of simply characteristic constant coefficient partial differential equations $P(D)u = f$. These estimates

$$||u||_{L^2(D_r)} \leq C\sqrt{d_r d_s}||f||_{L^2(D_s)}$$

depend on geometric quantities — the diameters $d_r$ and $d_s$ of the regions $D_r$, where we estimate $u$, and $D_s$, the support of $f$ — rather than weights. As these geometric quantities transform simply under translations, rotations, and dilations, the corresponding estimates share the same properties. In particular, this implies that they transform appropriately under change of units, and therefore are physically meaningful. The explicit dependence on the diameters implies the correct global growth estimates. The weighted $L^2$ estimates first proved by Agmon [1] in order to construct the generalized eigenfunctions for Laplacian plus potential in $\mathbb{R}^n$, and the more general and precise Besov type estimates of Agmon and Hörmander [2], are all simple direct corollaries of the estimate above.


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## 1. Introduction

Constant coefficient partial differential equations are translation invariant, so it is natural to seek estimates that share this property. For the Helmholtz equation, and other equations related to wave phenomena, $L^2$-norms are appropriate in bounded regions because they measure energy. For problems in all of $\mathbb{R}^n$, however, a solution with finite $L^2$-norm may radiate infinite power,[2] and therefore not satisfy the necessary physical constraints. The solution provided by Agmon [1] was to introduce $L^2_\delta$ spaces where weights $(1+|x|^2)^{\frac{\delta}{2}}$ correctly enforced the finite transmission of power, but gave up the translation invariance, as well as scaling properties necessary for the estimates to make sense in physical units. Later work by Agmon and Hörmander [2] used Besov spaces to exactly characterize solutions that radiated finite power, but these spaces also relied on a weight and therefore broke the translation invariance that is intrinsically associated with both the physics and the mathematics of the underlying problem. Later work by Kenig, Ponce, and Vega [9] modified the Agmon–Hörmander norms to regain better scaling properties.

Our goal here is to offer $L^2$ estimates that enforce finite radiation of power without using weights that destroy translation invariance and scaling properties. The following theorem, which applies to a class of scalar pde's with constant coefficients and simple characteristics, summarizes our main results, which will be proved as Theorem 5.1 and Theorem 6.1.

**Theorem.** *Let $P(D)$ be a constant coefficient partial differential operator on $\mathbb{R}^n$. Assume that it is either*

1. *real, of second order, and with no real double characteristics, or*
2. *of $N$-th order, $N \geqslant 1$, with admissible symbol (Definition 6.10) and no complex double characteristics.*

*Then there exists a constant $C(P,n)$ such that, for every open bounded $D_s \subset \mathbb{R}^n$, and every $f \in L^2(D_s)$, there is a $u \in L^2_{loc}(\mathbb{R}^n)$ satisfying*

$$P(D)u = f$$

*and for any bounded domain $D_r \subset \mathbb{R}^n$*

$$||u||_{L^2(D_r)} \leq C\sqrt{d_r d_s}||f||_{L^2(D_s)} \tag{1.1}$$

*where $d_j$ is the diameter of $D_j$, the supremum over all lines of the length of the intersection of the line with $D_j$; i.e.*

$$d_j = \sup_{\text{lines } l} \mu_1(l \cap D_j).$$

[2] Finite radiated power typically means that solutions decay fast enough at infinity. For outgoing solutions to the Helmholtz equation, radiated power can be expressed as the limit as $R \to \infty$ of the $L^2$ norm of the restriction of the solution to the sphere of radius $R$. It remains finite as long as solutions decay as $r^{-\frac{n-1}{2}}$ in $n$ dimensions.

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