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Large conformal metrics with prescribed scalar curvature [☆]

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Abstract

Let (M, g) be an n-dimensional compact Riemannian manifold. Let h be a smooth function on M and assume that it has a critical point $\xi \in M$ such that $h(\xi) = 0$ and which satisfies a suitable flatness assumption. We are interested in finding conformal metrics $g_\lambda = u_\lambda^{\frac{4}{n-2}} g$, with $u > 0$, whose scalar curvature is the prescribed function $h_\lambda := \lambda^2 + h$, where λ is a small parameter.

In the positive case, i.e. when the scalar curvature R_g is strictly positive, we find a family of "bubbling" metrics g_λ, where u_λ blows up at the point ξ and approaches zero far from ξ as λ goes to zero.

In the general case, if in addition we assume that there exists a non-degenerate conformal metric $g_0 = u_0^{\frac{4}{n-2}} g$, with $u_0 > 0$, whose scalar curvature is equal to h, then there exists a bounded family of conformal metrics $g_{0,\lambda} = u_{0,\lambda}^{\frac{4}{n-2}} g$, with $u_{0,\lambda} > 0$, which satisfies $u_{0,\lambda} \to u_0$ uniformly as $\lambda \to 0$. Here, we build a second family of "bubbling" metrics g_λ, where u_λ blows up at the point ξ and approaches u_0 far from ξ as λ goes to zero. In particular, this shows that this problem admits more than one solution.

MSC: 35J60; 53C21

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1. Introduction

Let (M, g) be a smooth compact manifold without boundary of dimension $n \geq 3$. The prescribed scalar curvature problem (with conformal change of metric) is

given a function h on M does there exist a metric $\tilde{g}$ conformal to g such that the scalar curvature of $\tilde{g}$ equals h?

Given a metric $\tilde{g}$ conformal to g, i.e. $\tilde{g} = u^{\frac{4}{n-2}} g$ for some smooth conformal factor $u > 0$, this problem is equivalent to finding a solution to

$$-\Delta_g u + c(n) R_g u = c(n) h u^p, \quad u > 0 \quad \text{on } M,$$

where $\Delta_g = \mathrm{div}_g \nabla_g$ is the Laplace–Beltrami operator, $c(n) = \frac{n-2}{4(n-1)}$, $p = \frac{n+2}{n-2}$, and R_g denotes the scalar curvature associated to the metric g. Since it does not change the problem, we replace $c(n)h$ with h. We are then led to study the problem

$$-\Delta_g u + c(n) R_g u = h u^p, \quad u > 0 \quad \text{on } M. \tag{1.1}$$

We suppose that h is not constant, otherwise we would be in the special case of the Yamabe problem which has been completely solved in the works by Yamabe [34], Trudinger [32], Aubin [3], and Schoen [29]. For this reason we can assume in (1.1) that R_g is a constant.

In the book [5, Chapter 6], Aubin gives an exhaustive description of known results. Next, we briefly recall some of them.

- **The negative case, i.e. $R_g < 0$.**

A necessary condition for existence is that $\int_M h d\nu_g < 0$ (a more general result can be found in [20]).

When $h < 0$, (1.1) has a unique solution (see for instance [20,3]). The situation turns out to be more complicated when h vanishes somewhere on M or if it changes sign. When $\max_M h = 0$, Kazdan and Warner [20], Ouyang [25], Vázquez and Véron [33], and del Pino [10] proved the existence of a unique solution, provided that a lower bound on R_g, depending on the zero set of h, is satisfied. The general case was studied by Rauzy in [26], who extended the previous results to the case when h changes sign. Letting $h^- := \min\{h, 0\}$ and $h^+ := \max\{h, 0\}$, the theorem proved in [26] reads as follows.

Theorem 1.1. *Let* $\mathcal{A} := \left\{ u \in H_g^1(M) \mid u \geq 0,\ u \not\equiv 0,\ \int_M h^- u d\nu_g = 0 \right\}$ *and*

$$\Lambda_0 := \inf_{u \in \mathcal{A}} \frac{\int_M |\nabla u|^2 d\nu_g}{\int_M u^2 d\nu_g},$$

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