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Blowup solutions for a nonlinear heat equation involving a critical power nonlinear gradient term

Tej-Eddine Ghoul [a], Van Tien Nguyen [a,*], Hatem Zaag [b,1]

[a] *New York University in Abu Dhabi, Departement of Mathematics, Computational Research Building A2, Saadiyat Island, P.O. Box 129188, Abu Dhabi, United Arab Emirates*

[b] *Université Paris 13, Sorbonne Paris Cité, Institut Galilée, LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France*

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Abstract

We consider the following exponential reaction–diffusion equation involving a nonlinear gradient term:

$$\partial_t U = \Delta U + \alpha|\nabla U|^2 + e^U, \quad (x,t) \in \mathbb{R}^N \times [0,T), \quad \alpha > -1.$$

We construct for this equation a solution which blows up in finite time $T > 0$ and satisfies some prescribed asymptotic behavior. We also show that the constructed solution and its gradient blow up in finite time T simultaneously at the origin, and find precisely a description of its final blowup profile. It happens that the quadratic gradient term is critical in some sense, resulting in the change of the final blowup profile in comparison with the case $\alpha = 0$. The proof of the construction is inspired by the method of Merle and Zaag in 1997. It relies on the reduction of the problem to a finite dimensional one, and uses the index theory to conclude. One of the major difficulties arising in the proof is that outside the *blowup region*, the spectrum of the linearized operator around the profile can never be made negative. Truly new ideas are needed to achieve the control of the outer part of the solution. Thanks to a geometrical interpretation of the parameters of the finite dimensional problem in terms of the blowup time and the blowup point, we obtain the stability of the constructed solution with respect to perturbations in the initial data.

* Corresponding author.
E-mail addresses: Teg6@nyu.edu (T.-E. Ghoul), Tien.Nguyen@nyu.edu (V.T. Nguyen), Hatem.Zaag@univ-paris13.fr (H. Zaag).

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1. Introduction

We are interested in the following nonlinear heat equation:

$$\begin{cases} \partial_t U & = & \Delta U + \alpha |\nabla U|^r + e^U, \\ U(0) & = & U_0, \end{cases} \tag{1.1}$$

where $U(t) : x \in \mathbb{R}^N \to U(x,t) \in \mathbb{R}$, Δ and ∇ stand for the Laplacian and the gradient in $\mathbb{R}^N$ with $N \geq 1$,

$$r = 2 \quad \text{and} \quad \alpha > -1.$$

Equation (1.1) can be viewed as the limiting case as $p \to +\infty$, for the following critical equation which was introduced by Chipot and Weissler [7]:

$$\partial_t U = \Delta U + \alpha |\nabla U|^r + |U|^{p-1} U, \quad \text{with } p > 1 \text{ and } r = \frac{2p}{p+1}. \tag{1.2}$$

The Cauchy problem for (1.1) can be solved in several functional spaces $\mathcal{F}$, for example $\mathcal{F} = W^{1,\infty}(\mathbb{R}^N)$ or in a special affine space $\mathcal{F} = \mathcal{H}_a$ for some positive constant a with

$$\mathcal{H}_a = \{u \in \psi + W^{1,\infty}(\mathbb{R}^N) \text{ with } \psi(x) = -\ln(1 + a|x|^2)\}. \tag{1.3}$$

In particular, the problem (1.1) has a unique classical solution $U(t) \in \mathcal{F}$ defined on $[0, T)$ with $T \leq +\infty$ (see Remark 1.3 for more details). In the case $T = +\infty$, $U(t)$ is a global solution for (1.1); on the contrary, i.e. $T < +\infty$, we say that the solution $U(t)$ blows up in finite time T, in the sense that

$$\begin{aligned} &\lim_{t \to T} \|U(t)\|_{W^{1,\infty}(\mathbb{R}^N)} = +\infty, \\ \text{or } &\lim_{t \to T} \|U(t) - \psi\|_{W^{1,\infty}(\mathbb{R}^N)} = +\infty \text{ in the second case.} \end{aligned}$$

As for equation (1.1), the value $r = 2$ is a critical exponent for different reasons ($r < 2$ and $r > 2$ correspond to the subcritical and supercritical cases). One reason is that, when $r = 2$, equation (1.1) is invariant under the following transformation:

$$\forall \lambda > 0, \quad U_\lambda(x,t) = 2 \ln \lambda + U(\lambda x, \lambda^2 t), \tag{1.4}$$

as for the equation without the gradient term, i.e. $\alpha = 0$. Let us recall that equation (1.2) is invariant under the transformation

$$\forall \lambda > 0, \quad U_\lambda(x,t) = \lambda^{\frac{2}{p-1}} U(\lambda x, \lambda^2 t). \tag{1.5}$$