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Boundary layer problem and zero viscosity-diffusion limit of the incompressible magnetohydrodynamic system with no-slip boundary conditions

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Abstract

The purpose of this paper is to study the boundary layer problem and zero viscosity-diffusion limit of the initial boundary value problem for the incompressible viscous and diffusive magnetohydrodynamic (MHD) system with (no-slip characteristic) Dirichlet boundary conditions and to prove that the corresponding Prandtl's type boundary layer are stable with respect to small viscosity-diffusion coefficients. The main difficulty here comes from Dirichlet boundary conditions for the velocity and magnetic field. Firstly, we identify a non-trivial class of initial data for which we can establish the uniform stability of the Prandtl's type boundary layers and prove rigorously the solution of incompressible viscous-diffusion MHD system converges strongly to the sum of the solution to the ideal MHD system and the approximating solution to Prandtl's type boundary layer equation by using the elaborate energy methods and the special structure of the solution to ideal MHD system with the initial data we identify here, which yields that there exists the cancellation between the boundary layer of the velocity and that of the magnetic field. Secondly, for general initial data, we obtain zero viscosity-diffusion limit of the incompressible viscous and diffusive MHD system with the different horizontal and vertical viscosities and magnetic diffusions, when they go to zero with the different speeds, and, we prove rigorously the convergence of the incompressible viscous and diffusion MHD system to the ideal MHD system or the anisotropic MHD system by constructing the exact boundary

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layers and using the elaborate energy methods. We also mention that these results obtained here should be the first rigorous ones on the stability of Prandtl's type boundary layer for the incompressible viscous and diffusion MHD system with no-slip characteristic boundary condition.

MSC: 35Q30; 35Q35; 35Q40; 35Q53; 35Q55; 76D05; 76Y07

Keywords: Incompressible viscous and diffusive MHD system; Ideal MHD system; Anisotropic inviscid MHD system; Boundary layer for no-slip boundary conditions

1. Introduction

We consider in this paper zero viscosity and diffusion limit for the three-dimensional incompressible viscous and diffusive magnetohydrodynamic(MHD) system with Dirichlet (no-slip characteristic) boundary conditions

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon - \varepsilon \Delta u^\varepsilon = b^\varepsilon \cdot \nabla b^\varepsilon, \text{ in } \Omega \times (0,T), \tag{1.1}$$

$$\partial_t b^\varepsilon + u^\varepsilon \cdot \nabla b^\varepsilon - \varepsilon \Delta b^\varepsilon = b^\varepsilon \cdot \nabla u^\varepsilon, \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$div u^\varepsilon = 0, div b^\varepsilon = 0, \quad \text{in } \Omega \times (0,T), \tag{1.3}$$

$$u^\varepsilon = 0, b^\varepsilon = 0, \quad \text{in } \partial\Omega \times (0,T), \tag{1.4}$$

$$u^\varepsilon(t=0) = u_0^\varepsilon, b^\varepsilon(t=0) = b_0^\varepsilon, \text{ with } \quad div u_0^\varepsilon = div b_0^\varepsilon = 0 \quad \text{on} \quad \Omega \tag{1.5}$$

and

$$\partial_t u^{\eta,\nu} + u^{\eta,\nu} \cdot \nabla u^{\eta,\nu} - \nu_1 \partial_z^2 u^{\eta,\nu} - \eta_1 \Delta_{x,y} u^{\eta,\nu} + \nabla p^{\eta,\nu}$$
$$= b^{\eta,\nu} \cdot \nabla b^{\eta,\nu}, \text{ in } \Omega \times (0,T), \tag{1.6}$$

$$\partial_t b^{\eta,\nu} + u^{\eta,\nu} \cdot \nabla b^{\eta,\nu} - \nu_2 \partial_z^2 b^{\eta,\nu} - \eta_2 \Delta_{x,y} b^{\eta,\nu} = b^{\eta,\nu} \cdot \nabla u^{\eta,\nu}, \quad \text{in } \Omega \times (0,T), \tag{1.7}$$

$$div u^{\eta,\nu} = 0, div b^{\eta,\nu} = 0, \quad \text{in } \Omega \times (0,T), \tag{1.8}$$

$$u^{\eta,\nu} = 0, b^{\eta,\nu} = 0, \quad \text{in } \partial\Omega \times (0,T), \tag{1.9}$$

$$u^{\eta,\nu}(t=0) = u_0^{\eta,\nu}, b^{\eta,\nu}(t=0) = b_0^{\eta,\nu}, \text{ with } div u_0^{\eta,\nu} = div b_0^{\eta,\nu} = 0 \quad \text{on } \Omega, \tag{1.10}$$

where $\Omega = \omega \times [0,h]$ or $\Omega = \omega \times (0,\infty)$ with $\omega = \mathbb{T}^2$ or $\mathbb{R}^2$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional Laplace operator, ε is the viscosity and magnetic diffusion coefficient. Also, $\Delta_{x,y}$ denotes the two-dimensional Laplace operator in the variables x and y. The constants ν_1 or ν_2 and η_1 or η_2 represent, respectively, the vertical and horizontal viscosities or magnetic diffusions. $u^\varepsilon, p^\varepsilon, b^\varepsilon$ or $u^{\eta,\nu}, p^{\eta,\nu}, b^{\eta,\nu}$ are the velocity, the pressure and the magnetic field of MHD. Here $\eta = (\eta_1, \eta_2)$ and $\nu = (\nu_1, \nu_2)$.

The well-posedness, regularity and asymptotic limit problem on the incompressible viscous and diffusive MHD system (1.1)–(1.3) and (1.6)–(1.8) in the whole space or with slip/no-slip boundary conditions have been studied extensively, see [2–5,9,10,19,22–25] and therein references. When $\eta_1 > 0, \eta_2 > 0, \nu_1 > 0$ and $\nu_2 > 0$, the MHD system in the whole space and in the

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