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Nonuniform dependence on initial data for compressible gas dynamics: The periodic Cauchy problem

B.L. Keyfitz [a,*], F. Tığlay [b,*]

[a] Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210-1174, United States

[b] Ohio State University, Newark, 1179 University Drive, Newark, OH 43055, United States

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Abstract

We start with the classic result that the Cauchy problem for ideal compressible gas dynamics is locally well posed in time in the sense of Hadamard; there is a unique solution that depends continuously on initial data in Sobolev space H^s for $s > d/2 + 1$ where d is the space dimension. We prove that the data to solution map for periodic data in two dimensions although continuous is not uniformly continuous.

Keywords: Compressible gas dynamics; Well-posedness; Hyperbolic conservation laws

0. Introduction

The compressible gas dynamics equations of ideal hydrodynamics are given by the system

$$\begin{aligned} \rho_t + \nabla\cdot(\rho\mathbf{u}) &= 0 \\ (\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p &= 0 \\ (E\rho)_t + \nabla\cdot(E\rho\mathbf{u} + p\mathbf{u}) &= 0 \end{aligned} \tag{1}$$

* Corresponding authors.
E-mail addresses: bkeyfitz@math.ohio-state.edu (B.L. Keyfitz), tiglay.1@osu.edu (F. Tığlay).

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with $E = e + \frac{1}{2}|\mathbf{u}|^2$ the total energy and $e = \frac{p}{(\gamma - 1)\rho}$ the internal energy, expressed in terms of density ρ, pressure p and velocity $\mathbf{u}$.

Classical solutions and well-posedness in Sobolev spaces (existence and uniqueness of solutions as well as continuous dependence of solutions on initial data) of the initial value problem for (1) have been studied extensively, see for instance [9,14–16]. Sobolev space results are all local in time. In one space dimension shock waves form in finite time for almost all data in H^s, and for later times only weak solutions exist. (The definition of weak solutions, and well-posedness theory in $BV_{\text{loc}} \cap L^1_{\text{loc}}$, which are not the subject of this paper, can be found in [2] and [3].) In higher dimensions there is as yet no existence theory for weak solutions, and classical (Sobolev space) solutions have a finite-time life span for almost all data [14,16].

Our goal is to study continuity properties of the solution map for classical solutions; in this paper we prove that for periodic data the initial-data to solution map is not uniformly continuous in Sobolev spaces. In a companion paper, [8], we extend this result to H^s data in the plane. Throughout, we assume s to be large enough for classical results to hold.

We consider solutions $U = U(\mathbf{x}, t)$ that take values in a compact subset of the state space $G = \{U \equiv (\rho, \mathbf{u}, p) \mid \rho, p > 0\}$, defined as the region where the physical quantities ρ and e are positive, and the system is symmetrizable hyperbolic.

In two dimensions, since we are considering classical solutions, we can ignore conservation form and write system (1) as

$$\begin{aligned} \rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) &= 0 \\ u_t + uu_x + vu_y + h_x + \frac{h}{\rho}\rho_x &= 0 \\ v_t + uv_x + vv_y + h_y + \frac{h}{\rho}\rho_y &= 0 \\ h_t + uh_x + vh_y + (\gamma - 1)h(u_x + v_y) &= 0\,. \end{aligned} \tag{2}$$

The parameter γ denotes the ratio of specific heats (typically $1 < \gamma < 3$) and $h = p/\rho = (\gamma - 1)e$ is a multiple of the internal energy.

We study this system in Sobolev spaces on the two dimensional torus: $H^s(\mathbb{T}^2)$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The Sobolev norm is given by

$$\|u\|_s^2 = \langle \Lambda^s u, \Lambda^s u \rangle\,,$$

where $\Lambda^s = (1 - \Delta)^{s/2}$ and $\langle\ ,\ \rangle$ denotes the L^2 inner product. Defining $U = (\rho, u, v, h)$ and $U(t) = U(\cdot, t)$, our main result is

Theorem 1 *(Nonuniform dependence on initial data). For $s > 2$, the data to solution map $U(0) \to U(t)$ for the system (2) is not uniformly continuous from a closed ball centered at $(\rho_0, 0, 0, h_0)$ in $(H^s(\mathbb{T}^2))^4$ into $C([0, T]; (H^s(\mathbb{T}^2))^4)$.*

We note the significance of $s > 2$. The well-posedness theory for symmetrizable hyperbolic systems, which forms the basis for our analysis, is credited to Gårding [4], Leray [12], Kato [9] and Lax [11]. Solutions for quasilinear systems in d space dimensions exist in spaces H^s for