



Local existence and uniqueness of strong solutions to the Navier–Stokes equations with nonnegative density

Jinkai Li

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China

Received 12 April 2016; revised 15 June 2017

Abstract

In this paper, we consider the initial-boundary value problem to the nonhomogeneous incompressible Navier–Stokes equations. Local strong solutions are established, for any initial data $(\rho_0, u_0) \in (W^{1,\gamma} \cap L^\infty) \times H_{0,\sigma}^1$, with $\gamma > 1$, and if $\gamma \geq 2$, then the strong solution is unique. The initial density is allowed to be nonnegative, and in particular, the initial vacuum is allowed. The assumption on the initial data is weaker than the previous widely used one that $(\rho_0, u_0) \in (H^1 \cap L^\infty) \times (H_{0,\sigma}^1 \cap H^2)$, and no compatibility condition is required.

© 2017 Elsevier Inc. All rights reserved.

MSC: 76D03; 76D05

Keywords: Local existence and uniqueness; Density-dependent incompressible Navier–Stokes equations; Compatibility condition; Gronwall type inequality

1. Introduction

The motion of the incompressible fluid in a domain Ω is governed by the following nonhomogeneous incompressible Navier–Stokes equations

$$\partial_t \rho + u \cdot \nabla \rho = 0, \quad (1.1)$$

$$\rho(\partial_t u + (u \cdot \nabla)u) - \Delta u + \nabla p = 0, \quad (1.2)$$

E-mail address: jklimath@gmail.com.

<http://dx.doi.org/10.1016/j.jde.2017.07.021>

0022-0396/© 2017 Elsevier Inc. All rights reserved.

$$\operatorname{div} u = 0, \quad (1.3)$$

in $\Omega \times (0, \infty)$, where the nonnegative function ρ is the density of the fluid, the vector field u denotes the velocity of the flow, and the scalar function p presents the pressure.

Since Leray's pioneer work [20] in 1934, in which he established the global existence of weak solutions to the homogeneous incompressible Navier–Stokes equations, i.e. system (1.1)–(1.3) with positive constant density, there has been a considerable number of papers devoted to the mathematical analysis on the incompressible Navier–Stokes equations. A generalization of Leray's result to the corresponding nonhomogeneous system, i.e. system (1.1)–(1.3) with variable density, was first made by Antontsev–Kazhikov in [3], for the case that the initial density is away from vacuum, see also the book Antontsev–Kazhikov–Monakhov [4]. For the case that the initial density is allowed to have vacuum, the global existence of weak solutions to system (1.1)–(1.3) was proved by Simon [29,30] and Lions [24]. However, the uniqueness and smoothness of weak solutions to the nonhomogeneous Navier–Stokes equation, even for the two dimensional case, is still an open problem; note that it is well known that weak solutions to the two dimensional homogeneous incompressible Navier–Stokes equations are unique, and are smooth immediately after the initial time, see, e.g., Ladyzhenskaya [18] and Temam [31].

Local existence (but without uniqueness) of strong solutions to the nonhomogeneous incompressible Navier–Stokes equations was first established by Antontsev–Kazhikov [3], under the assumption that the initial density is bounded and away from zero and the initial velocity has H^1 regularity. Local in time strong solutions, which enjoy the uniqueness, were later obtained by Ladyzhenskaya–Solonnikov [19], Padula [25,26] and Itoh–Tani [17]. Some more advances concerning the existence and uniqueness of strong solutions, in the framework of the so-called critical spaces, to the nonhomogeneous incompressible Navier–Stokes equations have been made recently, see, e.g., [1,2,9–12,15,27,28]. It should be mentioned that in all the works [1–3,9–12,15,17,19,25–28], the initial density is assumed to have positive lower bound, and thus no vacuum is allowed.

For the general case that the initial density is allowed to have vacuum, Choe–Kim [7] first proved the local existence and uniqueness of strong solutions to the initial-boundary value problem of system (1.1)–(1.3), with initial data (ρ_0, u_0) satisfying

$$0 \leq \rho_0 \in H^1 \cap L^\infty, \quad u_0 \in H^2 \cap H_{0,\sigma}^1, \quad (1.4)$$

and the compatibility condition

$$\Delta u_0 - \nabla p_0 = \sqrt{\rho_0} g, \quad (1.5)$$

for some $(p_0, g) \in H^1 \times L^2$. Since the work [7], conditions (1.4)–(1.5) and their necessary modifications are widely used, as the standard assumptions, in many papers concerning the studies of the existence and uniqueness of strong solutions, with initial vacuum allowed, to the nonhomogeneous Navier–Stokes equations and some related systems, such as the magnetohydrodynamics (MHD) and liquid crystals, see, e.g., [6,8,13,14,16,21,32,33].

Noticing that, when the initial vacuum is taken into consideration, conditions (1.4)–(1.5) are so widely used in the literatures to study the existence and uniqueness of strong solutions to the nonhomogeneous Navier–Stokes equations and some related models, we may ask if one can reduce the regularities on the initial data stated in (1.4) and drop the compatibility condition (1.5), so that the result of existence and uniqueness of strong solutions to the corresponding systems

Download English Version:

<https://daneshyari.com/en/article/5773993>

Download Persian Version:

<https://daneshyari.com/article/5773993>

[Daneshyari.com](https://daneshyari.com)