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Heat kernels for non-symmetric diffusion operators with jumps ☆

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Abstract

For $d \geqslant 2$, we prove the existence and uniqueness of heat kernels to the following time-dependent second order diffusion operator with jumps:

$$\mathscr{L}_t := \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{ij}^2 + \sum_{i=1}^{d} b_i(t,x)\partial_i + \mathscr{L}_t^{\kappa},$$

where $a = (a_{ij})$ is a uniformly bounded, elliptic, and Hölder continuous matrix-valued function, b belongs to some suitable Kato's class, and $\mathscr{L}_t^{\kappa}$ is a non-local α-stable-type operator with bounded kernel κ. Moreover, we establish sharp two-sided estimates, gradient estimate and fractional derivative estimate for the heat kernel under some mild conditions.

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1. Introduction

Let $C_0(\mathbb{R}^d)$ be the Banach space of all continuous functions on $\mathbb{R}^d$ vanishing at infinity equipped with uniform norm, and $C_c(\mathbb{R}^d)$ the space of all continuous functions on $\mathbb{R}^d$ with compact support. Let $\mathscr{L}$ be a linear operator on $C_0(\mathbb{R}^d)$ with domain $\mathrm{Dom}(\mathscr{L})$. Suppose that $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(\mathscr{L})$. We say $\mathscr{L}$ satisfies a positive maximum principle if for all $f \in C_c^\infty(\mathbb{R}^d)$ reaching a positive maximum at point $x_0 \in \mathbb{R}^d$, then $\mathscr{L} f(x_0) \leqslant 0$. The well-known Courrège theorem (cf. [1, Theorem 3.5.3]) states that $\mathscr{L}$ satisfies the positive maximum principle if and only if $\mathscr{L}$ takes the following form

$$\begin{aligned}\mathscr{L} f(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\partial_{ij}^2 f(x) + \sum_{i=1}^d b_i(x)\partial_i f(x) + c(x)f(x) \\ + \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - 1_{\{|z|\leqslant 1\}} z\cdot\nabla f(x)\right)\mu_x(\mathrm{d}z),\end{aligned} \tag{1.1}$$

where $a = (a_{ij}(x))_{1\leqslant i,j\leqslant d}$ is a $d \times d$-symmetric positive definite matrix-valued measurable function on $\mathbb{R}^d$, $b(x): \mathbb{R}^d \to \mathbb{R}^d$, $c: \mathbb{R}^d \to (-\infty, 0]$ are measurable functions and $\mu_x(\mathrm{d}z)$ is a family of Lévy measures, with that a, b, c, μ enjoy some continuity with respect to x (see [22]). On the other hand, from the probabilistic viewpoint, consider the following SDE with jumps:

$$\begin{aligned}\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t + \int_{|z|\leqslant 1} g(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) \\ + \int_{|z|>1} g(X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z), \quad X_0 = x,\end{aligned} \tag{1.2}$$

where $\sigma(x) = \sqrt{a(x)}$, $g(x,z): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, W is a d-dimensional standard Brownian motion, while N is a Poisson random measure with intensity measure ν, and $\tilde{N}$ is the associated compensated Poisson random measure. Under some Lipschitz assumptions in x-variable on $\sigma(x)$, $b(x)$ and $g(x,z)$, it is well known that the above SDE admits a unique strong solution, which defines a strong Markov process whose infinitesimal generator $\mathscr{L}$ is of the form (1.1) with $\mu_x(\mathrm{d}z) = \nu \circ g^{-1}(x,\cdot)(\mathrm{d}z)$ (see [21]). A natural question is whether SDE (1.2) has a (weak) solution without Lipschitz assumption on $\sigma(x)$, $b(x)$ and $g(x,z)$, and how about its density.

In this work we are concerned with the existence, uniqueness, and estimates of fundamental solutions of time-dependent version of the operator $\mathscr{L}$ in (1.1), with minimal regularity assumptions on $a(t,x)$, $b(t,x)$ and $\kappa(t,x,z)$, where $\kappa(t,x,z) := |z|^{d+\alpha}\mu_{t,x}(\mathrm{d}z)/\mathrm{d}z$. More precisely, we shall consider the following time-inhomogeneous and non-symmetric non-local operators:

$$\mathscr{L}_t f(x) := \mathscr{L}_t^a f(x) + b_t\cdot\nabla f(x) + \mathscr{L}_t^\kappa f(x), \tag{1.3}$$

where

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