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Liouville type theorem for some nonlocal elliptic equations

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Abstract

In this paper, we prove some Liouville theorem for the following elliptic equations involving nonlocal nonlinearity and nonlocal boundary value condition

$$\begin{cases} -\Delta u(y) = \int\limits_{\partial\mathbb{R}^N_+} \frac{F(u(x',0))}{|(x',0)-y|^{N-\alpha}} dx' g(u(y)), & y \in \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \nu}(x',0) = \int\limits_{\mathbb{R}^N_+} \frac{G(u(y))}{|(x',0)-y|^{N-\alpha}} dy f(u(x',0)), & (x',0) \in \partial\mathbb{R}^N_+, \end{cases}$$

where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$, f, g, F, G are some nonlinear functions. Under some assumptions on the nonlinear functions f, g, F, G, we will show that this equation doesn't possess nontrivial positive solution. We extend the Liouville theorems from local problems to nonlocal problem. We use the moving plane method to prove our result.

MSC: 35J60; 35J57; 35J15

Keywords: Liouville type theorem; Moving plane method; Maximum principle; Nonlocal problem

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1. Introduction

In the studying of the existence of solutions for non-variational elliptic equations on bounded domain, we usually use the topological methods such as the Leray–Schauder degree theory to get the existence results. In order to apply such a theory, a priori bound on the solution is usually needed. As far as we know, the blow-up method is the most powerful tool for proving a priori bound. The spirit of blow up method is straightforward. Suppose on the contrary that there exists a sequence of solutions $\{u_n\}$ with $M_n = u_n(x_n) = \|u_n\|_{L^\infty(\Omega)} \to \infty$, then we make a scaling on this sequence of solutions and get $v_n(x) = \frac{1}{M_n} u_n(M_n^k x + x_n)$ which is bounded. Hence, by the regularity theory of elliptic equations, we can assume that $v_n \to v$ in $C^{2,\gamma}_{loc}(\Omega_\infty)$ with $\|v\|_{L^\infty} = 1$ and satisfying some limit equation in Ω_∞, where either $\Omega_\infty = \mathbb{R}^N$ or $\Omega_\infty = \mathbb{R}^N_+$ depending on the speed of x_n goes to the boundary of Ω. On the other hand, if we can prove the limit equations don't possess nontrivial solutions, then we get a contradiction, hence the solutions of the original problem must be bounded. From the descriptions of the blow-up procedure, it is easy to see that a priori bound of elliptic equations on bounded domain is equivalent to the Liouville type theorems for the limit equations. Hence, from the past few decades, Liouville theorems for elliptic equations have attracted much attention of scientists and many results were obtained. The most remarkable result on this aspect is the results in [12], in which the authors studied the nonexistence results for the following elliptic equation

$$-\Delta u = u^p \quad \text{in} \quad \mathbb{R}^N, \quad u \geq 0. \tag{1.1}$$

The authors proved, among other things, that problem (1.1) does not possess positive solution provided $0 < p < \frac{N+2}{N-2}$. Moreover, this result is optimal in the sense that for any $p \geq \frac{N+2}{N-2}$, there are infinitely many positive solutions to problem (1.1). Thus the Sobolev exponent $\frac{N+2}{N-2}$ is the dividing exponent between existence and nonexistence of positive solutions. For this reason, the exponent $\frac{N+2}{N-2}$ is usually called the critical exponent for equation (1.1). Later, in order to get a priori bound for some non-variational elliptic equations on bounded domains, the authors also studied a similar problem in half space

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+ \end{cases} \tag{1.2}$$

in [13]. They proved that problem (1.2) also does not possess positive solution for $0 < p < \frac{N+2}{N-2}$. Later, in order to simplify the proofs in [13] and [12], W. Chen and C. Li proved similar results by using the moving plane method in [1]. The idea of [1] is very creative, they proved the solution is symmetric in every direction and with respect to every point, hence the solution must be a constant, finally, they deduced from the equation that the constant must be zero. After the results of [1], the moving plane method or its variant the moving sphere method were widely used in proving Liouville theorems for elliptic equations, see [7][8][11][14][15][16][18][30][31][32] and the references therein. At the same time, the moving plane method was also widely used in proving Liouville theorems for integral equations, integral systems and fractional Laplacian equations, we refer the readers to [2][3][4][5][6][10][17][19][20][28][29] for more details.

Among the above works, we should mention the paper [7]. In this paper, L. Damascelli and F. Gladiali studied the nonexistence of weak positive solution for the following nonlinear problem with general nonlinearity

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