Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 263 (2017) 6821–6843

Journal of Differential Equations

www.elsevier.com/locate/jde

Harnack inequality for nonlinear elliptic equations with strong absorption

Ahmed Mohammed [a,*], Giovanni Porru [b], Antonio Vitolo [c]

[a] *Department of Mathematical Sciences, Ball State University, Muncie, IN 47306, USA*
[b] *Dipartimento di Matematica e Informatica, Università di Cagliari, 09124, Cagliari, Italy*
[c] *Department of Civil Engineering, Università di Salerno, 84084 Fisciano, Salerno, Italy*

Received 16 October 2016; revised 15 March 2017
Available online 14 August 2017

Abstract

We develop Harnack inequalities for two different types of equations. First we consider a fully nonlinear uniformly elliptic equation related to the Pucci's maximal and the minimal operators. Next we consider a quasilinear equation related to the p-Laplacian. In both cases we consider lower order terms of Keller–Osserman type. Although the equations considered are quite different, we employ a unified method to approach both problems and the results we find are similar.

MSC: 35J60; 35B05; 35B40; 35D05

Keywords: Pucci extremal operators; Viscosity solutions; p-Laplace equations; Blow-up solutions; Harnack inequality

1. Introduction

In a recent paper [10], Martin Dindoš studied the Harnack inequality for non-negative classical solutions of $\Delta u = f(u)$ in domains in $\mathbb{R}^n$. In [10], Dindoš used a strict convexity condition and the Keller–Osserman condition on the nonlinear term f to obtain a global L^∞ estimate of all non-negative solutions to the aforementioned equation. The estimate was achieved by comparing nonnegative solutions to boundary blow-up solutions, the existence of which is assured by the

* Corresponding author.
E-mail addresses: amohammed@bsu.edu (A. Mohammed), porru@unica.it (G. Porru), vitolo@unisa.it (A. Vitolo).

http://dx.doi.org/10.1016/j.jde.2017.07.035
0022-0396/

Keller–Osserman condition. As an important ingredient Dindoš introduced a growth condition on f at infinity which, in conjunction with the global estimate, led to Harnack inequality for nonnegative solutions. The results in [10] extend the work of Finn and McOwen in [11].

A weakened version of the conditions used in [10] have been used in [16] to establish Harnack inequality for general second order uniformly elliptic equations in non-divergence form. In this paper we wish to extend Dindoš' work by not only replacing the principal operator with more general elliptic operators of two types, but also by significantly weakening some of the other conditions used in Dindoš' work. We use these conditions to develop Harnack inequality for non-negative viscosity solutions of fully nonlinear equations. The generalized Dindoš' condition with $p > 1$ allows to extend further the result to operators with different homogeneity degree in the gradient such as the p-Laplace operator. More specifically we first investigate the Harnack inequality for non-negative (viscosity) solutions of a fully nonlinear equation $H(x, u, Du, D^2u) = g(x, u)$, where $H(x, t, \xi, X)$ satisfies appropriate structural condition related to the Pucci extremal operators. Assuming that $g(x, t)$ satisfies $f(t) \leq g(x, t) \leq Tf(t)$ for some constant $T \geq 1$, and some non-negative function f that satisfies Dindoš' condition, we shall prove a Harnack inequality for non-negative viscosity solutions of the equations described above. For basic results on fully nonlinear equations we refer the reader to [3,7] and the references therein.

The second part of our investigation will focus on developing the Harnack inequality for non-negative weak solutions of the quasilinear equation $\operatorname{div}(|Du|^{p-2}Du) + b(x)u^{p-1} = g(x, u)$, where $p > 1$, and $g(x, t)$ satisfies the same condition as before but with f now satisfying a generalized Dindoš' condition (depending on $p > 1$). As pointed out earlier, the existence of weak boundary blow-up solutions of $\operatorname{div}(|Du|^{p-2}Du) = f(u)$ plays an important role in our approach. This infinite boundary value problem for such equations has been investigated by many authors. For the case $p = 2$ see [2,12,15] and references therein. For general $p > 1$ we refer to [9,13].

In recent years there has been considerable interest in absorption equations with nonlinear principal parts. The reader is referred to the interesting papers [4,5,8,14] and the references therein.

The paper is organized as follows. In Section 2, after introducing some basic facts on fully nonlinear uniformly elliptic equations, we derive Harnack inequality for non-negative viscosity solutions of differential inequalities involving the Pucci extremal operators with lower order terms. This, which is of independent interest in itself, would serve as the basic tool for proving our main Harnack inequality. We then establish the existence of viscosity supersolutions to Pucci maximal operators with lower order terms with nonlinear terms satisfying the Keller–Osserman condition. These supersolutions are used to develop a uniform global L^∞ estimate for all non-negative solutions to such equations. The Dindoš' condition, together with the above mentioned results, provide the necessary tools to derive the desired Harnack Inequality, Theorem 2.8.

In Section 3, we look at a class of quasilinear equations and recall some basic results about them that will aid in our study of the Harnack inequality. Next, we introduce a general version of the Dindoš' condition that is suited to the study of Harnack inequality of quasilinear equations. This section follows the same general approach of Section 2 to develop the Harnack inequality, Theorem 3.8, for non-negative weak solutions of the quasilinear equations under consideration.

In both Sections 2 and 3, we employ a useful estimate involving the nonlinear term f to derive the Harnack inequality. This estimate is proved in Appendix A as a consequence of the generalized Dindoš condition.