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Motion of a vortex filament on a slanted plane

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Abstract

We consider a nonlinear model equation, known as the Localized Induction Equation, describing the motion of a vortex filament immersed in an incompressible and inviscid fluid. We prove the unique solvability of an initial–boundary value problem describing the motion of a vortex filament on a slanted plane.

1. Introduction and problem setting

A vortex filament is a space curve on which the vorticity of the fluid is concentrated. Vortex filaments are used to model very thin vortex structures such as vortices that trail off airplane wings or propellers. In this paper, we prove the solvability of the following initial–boundary value problem which describes the motion of a vortex filament moving on a slanted plane.

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, & s \in I,\ t > 0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s \in I,\ t > 0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{a},\ \boldsymbol{x}_s(1,t) = \boldsymbol{e}_3, & t > 0, \end{cases} \tag{1.1}$$

where $\boldsymbol{x}(s,t) = {}^t(x_1(s,t), x_2(s,t), x_3(s,t))$ is the position vector of the vortex filament parametrized by its arc length s at time t, $\times$ is the exterior product in the three dimensional Euclidean space, $I = (0,1) \subset \mathbf{R}$ is an open interval, $\boldsymbol{a} \in \mathbf{R}^3$ is an arbitrary vector satisfying

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$|\boldsymbol{a}| = 1$, $\boldsymbol{e}_3 = {}^t(0, 0, 1)$, and subscripts s and t are differentiations with the respective variables. The length of the interval I is the length of the initial filament, which we set to 1 for simplicity since the proof is the same for filaments with general length. Problem (1.1) describes the motion of a segment of a vortex filament moving on a slanted plane. We can see that, by taking the trace $s = 0$ in the equation of problem (1.1), a filament moving according to problem (1.1) satisfies

$$\boldsymbol{x}_t(0, t) = \boldsymbol{a} \times \boldsymbol{x}_{ss}(0, t),$$

hence the end-point $\boldsymbol{x}(0, t)$ of the filament moves along the plane perpendicular to $\boldsymbol{a}$. The reason we also impose a boundary condition at $s = 1$ is for the following reason. A more intuitive problem setting for a vortex filament moving on a plane would be

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, & s > 0,\ t > 0, \\ \boldsymbol{x}(s, 0) = \boldsymbol{x}_0(s), & s > 0,\ t > 0, \\ \boldsymbol{x}_s(0, t) = \boldsymbol{a}, & t > 0, \end{cases} \tag{1.2}$$

which is a problem describing an infinitely long filament with one end moving along the plane perpendicular to $\boldsymbol{a}$. The solvability of problem (1.2) is a direct consequence of a previous work by the author and Iguchi [1], which proved the solvability of problem (1.2) with $\boldsymbol{a} = \boldsymbol{e}_3$, because the solution of problem (1.2) can be obtained by rotating the solution obtained in [1] in a way that $\boldsymbol{a}$ is transformed to $\boldsymbol{e}_3$. Hence, problem (1.2) for general $\boldsymbol{a}$ is essentially the same as the case $\boldsymbol{a} = \boldsymbol{e}_3$. So to describe the motion of a vortex filament on a slanted plane, we imposed a boundary condition at $s = 1$ to set a reference plane which allows us to express the slanted-ness of the plane that the filament is moving on. The motivation for considering problem (1.1) comes from the following problem.

$$\begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}, & s > 0,\ t > 0, \\ \boldsymbol{x}(s, 0) = \boldsymbol{x}_0(s), & s > 0, \\ \boldsymbol{x}_s(0, t) = \dfrac{\nabla B(\boldsymbol{x}(0, t))}{|\nabla B(\boldsymbol{x}(0, t))|}, & t > 0, \end{cases} \tag{1.3}$$

where $\nabla = {}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and for ${}^t(x_1, x_2, x_3) \in \mathbf{R}^3$, $B : \mathbf{R}^3 \to \mathbf{R}$ is a given function of the form

$$B(x_1, x_2, x_3) = x_3 - b(x_2, x_3)$$

for a given scalar function $b : \mathbf{R}^2 \to \mathbf{R}$. Problem (1.3) describes an infinitely long vortex filament moving on a surface given as the graph of b in the three-dimensional Euclidean space. Problem (1.3) is a generalization of the problem setting in [1] and can be seen as a simplified model for the motion of a tornado, where the ground is given by the graph of b, but the solvability for a general b seems hard, and as a first step, we chose the special case where the ground is a slanted plane.

The equation in problem (1.1) is called the Localized Induction Equation (LIE) which is derived by applying the so-called localized induction approximation to the Biot–Savart integral. The LIE was first derived by Da Rios [5] in 1906 and was re-derived twice independently by

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