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# Symmetry-breaking bifurcation for the one-dimensional Liouville type equation

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Dedicated to Professor Eiji Yanagida on the occasion of his sixtieth birthday

**Abstract**

The two-point boundary value problem for the one-dimensional Liouville type equation

$$\begin{cases} u'' + \lambda|x|^l e^u = 0, & x \in (-1,1), \\ u(-1) = u(1) = 0 \end{cases}$$

is considered, where $\lambda > 0$ and $l > 0$. In this paper, a symmetry-breaking result is obtained by using the Morse index. The problem

$$\begin{cases} u'' + \lambda|x|^l (u+1)^p = 0, & x \in (-1,1), \\ u(-1) = u(1) = 0 \end{cases}$$

is also considered, where $\lambda > 0$, $l > 0$, $p > 1$ and $(p-1)l > 4$.


*MSC:* 35B32; 35J25; 35J60; 34B18

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## 1. Introduction

In this paper, we consider the two-point boundary value problem for the one-dimensional Liouville type equation

$$\begin{cases} u'' + \lambda |x|^l e^u = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $l > 0$.

Jacobsen and Schmitt [7] presented the exact multiplicity result of radial solutions for the multi-dimensional problem

$$\begin{cases} \Delta u + \lambda |x|^l e^u = 0 & \text{in } B, \\ \qquad u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $l \geq 0$, $B := \{x \in \mathbf{R}^N : |x| < 1\}$ and $N \geq 1$. In the case $N = 1$, problem (1.2) is reduced to (1.1). We note here that every solution of (1.2) is positive in $B$, by the strong maximum principle. Jacobsen and Schmitt [7] proved the following (i)–(iii):

(i) if $1 \leq N \leq 2$, then there exists $\lambda_* > 0$ such that (1.2) has exactly two radial solutions for $0 < \lambda < \lambda_*$, a unique radial solution for $\lambda = \lambda_*$ and no radial solution for $\lambda > \lambda_*$;
(ii) if $3 \leq N < 10 + 4l$, then (1.2) has infinitely many radial solutions when $\lambda = (l + 2)(N - 2)$ and a finite but large number of radial solutions when $|\lambda - (l + 2)(N - 2)|$ is sufficiently small;
(iii) if $N \geq 10 + 4l$, then (1.2) has a unique radial solution for $0 < \lambda < (l + 2)(N - 2)$ and no radial solution for $\lambda \geq (l + 2)(N - 2)$.

Recently, Korman [14] gave an alternative proof of (i)–(iii), and his method is very interesting and easy to understand it. Results (i)–(iii) were established by Liouville [16], Gel'fand [4], Joseph and Lundgren [8] for problem (1.2) with $l = 0$, that is,

$$\begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

when $\Omega = B$.

A celebrated theorem by Gidas, Ni and Nirenberg [5] shows that every positive solution of (1.3) is radially symmetric when $\Omega = B$. However, when $\Omega$ is an annulus $A := \{x \in \mathbf{R}^N : a < |x| < b\}$, $a > 0$, problem (1.3) may has non-radial solutions. Indeed, Nagasaki and Suzuki [18] found that large non-radial solutions of (1.3) when $N = 2$ and $\Omega = A$. More precisely, for each sufficiently large $\mu > 0$, there exist $\lambda > 0$ and a non-radial solution $u$ of (1.3) such that $\int_A e^u dx = \mu$ when $N = 2$ and $\Omega = A$. Lin [15] showed that (1.3) has infinitely many symmetry-breaking bifurcation points when $N = 2$ and $\Omega = A$. Dancer [2] proved that non-radial solution branches emanating from the symmetry-breaking bifurcation points found by Lin [15] are unbounded. Kan [9,10] considered (1.3) with $\Omega = A$ and $N = 2$ and investigated the structure of non-radial solutions bifurcating from radial solutions in the case where $a$ is sufficiently small.

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