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A classical Perron method for existence of smooth solutions to boundary value and obstacle problems for degenerate-elliptic operators via holomorphic maps ☆

Paul M.N. Feehan [a,b,*]

[a] *Department of Mathematics, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, United States*

[b] *Department of Mathematics, Columbia University, New York, NY 10027, United States*

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Abstract

We prove existence of solutions to boundary value problems and obstacle problems for degenerate-elliptic, linear, second-order partial differential operators with partial Dirichlet boundary conditions using a new version of the Perron method. The elliptic operators considered have a degeneracy along a portion of the domain boundary which is similar to the degeneracy of a model linear operator identified by Daskalopoulos and Hamilton [9] in their study of the porous medium equation or the degeneracy of the Heston operator [21] in mathematical finance. Existence of a solution to the partial Dirichlet problem on a half-ball, where the operator becomes degenerate on the flat boundary and a Dirichlet condition is only imposed on the spherical boundary, provides the key additional ingredient required for our Perron method. Surprisingly, proving existence of a solution to this partial Dirichlet problem with "mixed" boundary conditions on a half-ball is more challenging than one might expect. Due to the difficulty in developing a global Schauder estimate and due to compatibility conditions arising where the "degenerate" and "non-degenerate boundaries" touch, one cannot directly apply the continuity or approximate solution methods. However, in dimension two, there is a holomorphic map from the half-disk onto the infinite strip in the complex plane and one can extend this definition to higher dimensions to give a diffeomorphism from the half-ball onto the infinite "slab". The solution to the partial Dirichlet problem on the half-ball can thus be converted to a partial Dirichlet problem on the slab, albeit for an operator which now has exponentially growing coefficients. The

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* Correspondence to: Department of Mathematics, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, United States.

E-mail address: feehan@math.rutgers.edu.

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required Schauder regularity theory and existence of a solution to the partial Dirichlet problem on the slab can nevertheless be obtained using previous work of the author and C. Pop [16]. Our Perron method relies on weak and strong maximum principles for degenerate-elliptic operators, concepts of continuous subsolutions and supersolutions for boundary value and obstacle problems for degenerate-elliptic operators, and maximum and comparison principle estimates previously developed by the author [13].

MSC: primary 35J70, 35J86, 49J40, 35R35, 35R45; secondary 49J20, 60J60

Keywords: Comparison principle; Degenerate elliptic differential operator; Non-negative characteristic form; Obstacle problem; Schauder regularity theory; Viscosity solution

Contents

1. Introduction

Suppose $\mathscr{O} \subsetneqq \mathbb{H}$ is a domain (possibly unbounded) in the open upper half-space $\mathbb{H} := \mathbb{R}^{d-1} \times \mathbb{R}_+$, where $d \geq 2$ and $\mathbb{R}_+ := (0, \infty)$, and $\partial_1\mathscr{O} := \partial\mathscr{O} \cap \mathbb{H}$ is the portion of the boundary $\partial\mathscr{O}$ of $\mathscr{O}$ which lies in $\mathbb{H}$, and $\partial_0\mathscr{O}$ is the interior of $\partial\mathbb{H} \cap \partial\mathscr{O}$, where $\partial\mathbb{H} = \mathbb{R}^{d-1} \times \{0\}$ is the boundary of