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A modified equatorial β-plane approximation modelling nonlinear wave-current interactions

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Abstract

A modification of the standard geophysical equatorial β-plane model equations, incorporating a gravitational-correction term in the tangent plane approximation, is derived. We present an exact solution satisfying the modified equations, whose form is explicit in the Lagrangian framework, and which represents three-dimensional, nonlinear oceanic wave-current interactions. It is rigorously established, by way of analytical and degree-theoretical considerations, that the solution is dynamically possible, in the sense that the mapping it prescribes from Lagrangian to Eulerian coordinates is a global diffeomorphism.

MSC: 76B15; 74G05; 37N10

Keywords: Exact solution; Global diffeomorphism; Lagrangian variables; Wave-current interactions; β-Plane

1. Introduction

The modelling of geophysical fluid dynamics in the equatorial region is a highly complex subject of vast importance which has recently witnessed a number of interesting mathematical developments. Geophysical fluid dynamics is the study of fluid motion where the Earth's rotation plays a significant role in the resulting dynamics, and accordingly Coriolis forces are incorporated into the governing Euler equation. The ensuing governing equations are applicable for a wide range of oceanic and atmospheric flows [11,14,27], thereby encapsulating the necessarily

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high-level of mathematical sophistication required to model such a rich variety of phenomena. This level of complexity leads to an inherent mathematical intractability in the model equations, and in order to mitigate this one typically employs oceanographical considerations in order to derive simpler approximate models.

A classical example which is typically employed in oceanographic considerations is the β-plane approximation, whereby the earth's curved surface is approximated (locally) by a tangent plane. In the context of modelling equatorial flows this approach is applicable when we restrict our focus to regions of relatively small latitudinal variation (to the order of 2°) about the equator; physically, the equator acts as a natural waveguide leading to equatorially-trapped zonally propagating waves which decay exponentially away from the equator, cf. [12]. We note that there has been an abundance of recent mathematical progress in deriving, and analysing, exact solutions to the β-plane equations modelling equatorial oceanic water waves [3–6,13,15, 17–21]—an interesting reflection on the relevance of exact solutions in physical oceanography may be found in [9].

However, we remark that while the β-plane approximation is regarded as reasonable for large-scale oceanographical considerations, nevertheless from a mathematical modelling perspective it is lamentable that an appreciable level of mathematical detail and structure is lost from the model equations as a result of the 'flattening out' of the earth's surface. A number of interesting mathematical approaches have been recently instigated which aim to retain some of this structure in modelling equatorial water waves, cf. [7,8,10,16]. The primary aim of this paper is to address this matter with a view to retaining artefacts of the geometry of the earth's curvature by way of incorporating a gravitational-correction term into the standard β-tangent plane model, resulting in the modified governing equations (3).

Following the derivation in Section 2, we present a mapping (4) which we claim is an exact solution to the modified equations (3) representing three-dimensional, nonlinear wave-current interactions; the zonally-periodic wavelike term is equatorially-trapped (exhibiting exponentially strong meridional decay) and propagates eastwards above a flow which accommodates a depth-invariant mean current—either following or adverse—of any physically plausible (as defined by (5)) magnitude. In Section 3 we prove by direct computation that the mapping (4), explicit in terms of Lagrangian labelling variables, is compatible with the governing equations (3), and that it maps the Lagrangian labelling domain to a fluid domain bounded above by the free-surface interface. We note that while the underlying current in the exact solution (4) assumes an apparently simple form in the Lagrangian framework, it greatly increases the complexity, both mathematically and physically, of the resulting fluid motion [13,18] in the Eulerian setting.

From an oceanographic perspective, large-scale currents and wave-current interactions play a major role in the geophysical dynamics of the equatorial region [7,8,11,12,22]. Aside from being physical important [2,25], the consideration of underlying currents, and wave-current interactions, is a compelling subject in its own right from a purely mathematical viewpoint. The robustness of the modified governing equations in admitting such a general range of underlying currents in our exact solution is attributable precisely to the gravitational-correction terms, and contrasts strongly with the situation in [15] where the range of admissible adverse currents is greatly restricted.

We complete our analysis in Section 4 by employing analytical considerations to establish that (4) defines a local diffeomorphism from the Lagrangian labelling domain to the fluid domain, and that this mapping is globally injective. Further degree-theoretical considerations then enable us to prove that (4) is, in fact, a global diffeomorphism: these deliberations establish rigorously that the (highly physically-complex!) motion prescribed by the mapping (4), which represents three-

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