ARTICLE IN PRESS YJDEQ:8792

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

The Dirichlet problem in a planar domain with two moderately close holes

M. Dalla Riva [a,b,*], P. Musolino [b]

[a] *Department of Mathematics, The University of Tulsa, 800 South Tucker Drive, Tulsa, OK 74104, USA*
[b] *Department of Mathematics, Aberystwyth University, Ceredigion SY23 3BZ, Wales, UK*

Received 6 December 2016; revised 8 March 2017

Abstract

We investigate a Dirichlet problem for the Laplace equation in a domain of $\mathbb{R}^2$ with two small close holes. The domain is obtained by making in a bounded open set two perforations at distance $|\epsilon_1|$ one from the other and each one of size $|\epsilon_1\epsilon_2|$. In such a domain, we introduce a Dirichlet problem and we denote by $u_{\epsilon_1,\epsilon_2}$ its solution. We show that the dependence of $u_{\epsilon_1,\epsilon_2}$ upon (ϵ_1, ϵ_2) can be described in terms of real analytic maps of the pair (ϵ_1, ϵ_2) defined in an open neighbourhood of $(0, 0)$ and of logarithmic functions of ϵ_1 and ϵ_2. Then we study the asymptotic behaviour of $u_{\epsilon_1,\epsilon_2}$ as ϵ_1 and ϵ_2 tend to zero. We show that the first two terms of an asymptotic approximation can be computed only if we introduce a suitable relation between ϵ_1 and ϵ_2.

MSC: 35J25; 31B10; 45A05; 35B25; 35C20

Keywords: Dirichlet problem; Singularly perturbed perforated planar domain; Moderately close holes; Laplace operator; Real analytic continuation in Banach space; Asymptotic expansion

1. Introduction

The asymptotic analysis of elliptic boundary value problems in domains with many holes which collapse one to the other while shrinking their sizes is a topic of growing interest and

* Corresponding author.
E-mail addresses: matteo-dallariva@utulsa.edu (M. Dalla Riva), musolinopaolo@gmail.com (P. Musolino).

http://dx.doi.org/10.1016/j.jde.2017.04.006
0022-0396/

Please cite this article in press as: M. Dalla Riva, P. Musolino, The Dirichlet problem in a planar domain with two moderately close holes, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.04.006

several authors have recently proposed different techniques and points of view. We mention for example the method based on multiscale asymptotic expansions which have been used by Bonnaillie-Noël, Dambrine, Tordeux, and Vial [5,6], Bonnaillie-Noël and Dambrine [3], and Bonnaillie-Noël, Dambrine, and Lacave [4] to study problems with two moderately close holes, *i.e.*, problems with two holes whose mutual distance tends to zero while their size tends to zero at faster speed. The case when the number of holes is large has been considered by Maz'ya, Movchan, and Nieves in a series of papers where they propose a mesoscale approximation method to analyse problems for the Laplace operator and for the system of linear elasticity. We mention, for example, Maz'ya and Movchan [23,24], and Maz'ya, Movchan, and Nieves [25–28]. The mesoscale approximation method does not require any periodicity assumption. If instead the holes have a periodic structure, then one can resort to the large literature in homogenization theory, where, rather then aiming at obtaining asymptotic expansions, one typically characterizes the limit value of the solution of a perturbed problem as the solution of a limiting problem. We refer, for instance, to the seminal works of Bakhvalov and Panasenko [2], Cioranescu and Murat [8,9], and Marčenko and Khruslov [22] and to the more recent 'periodic unfolding method' used, *e.g.*, by Cioranescu, Damlamian, Donato, Griso, and Zaki [7]).

In this paper, we consider a Dirichlet problem for the Laplace equation in a planar domain with two small close holes. The method adopted is different from those mentioned above. Indeed, we follow the 'functional analytic approach' which has been proposed by Lanza de Cristoforis for the analysis of linear and nonlinear singular perturbation problems (see, *e.g.*, Lanza de Cristoforis [17,19,20]) and which allows the representation of the solution in terms of elementary functions and of real analytic maps of the singular perturbation parameters. One of the advantages of the method is that real analytic maps can be expanded into power series and thus, as a byproduct of our analysis, we can deduce fully justified asymptotic expansions for the solution with any order of approximation. Moreover, the coefficients of such expansions can be explicitly and constructively computed by solving certain systems of integral equations (as shown in [13]). This method has been exploited for the analysis of Laplace and Poisson problems in domains with small close holes in [11] and in [12], respectively. In both of these papers, the conditions on the boundaries of the holes are of Neumann type. Here, instead, we will study a problem with Dirichlet conditions and we will focus on the two-dimensional case. This case is more involved than the higher dimensional case or the Neumann condition case because of the logarithmic behaviour induced by the two-dimensional fundamental solution. As we shall see, such logarithmic behaviour will force the introduction of a specific relation between the size and the distance of the holes if we wish to pass from the representation of the solution in terms of analytic maps to the explicit computation of the first asymptotic approximation terms.

We now proceed to introduce our problem and we start by defining the geometric setting. We fix once for all a real number $\alpha \in]0, 1[$ and three sets Ω^o, Ω_1 and Ω_2 that satisfy the following condition:

$$
\begin{aligned}
&\Omega^o, \Omega_1 \text{ and } \Omega_2 \text{ are open bounded connected subsets of } \mathbb{R}^2\\
&\text{of class } C^{1,\alpha}, \text{ they contain the origin 0 of } \mathbb{R}^2 \text{ and}\\
&\text{they have connected boundaries } \partial\Omega^o, \partial\Omega_1, \text{ and } \partial\Omega_2.
\end{aligned}
$$

Here the letter 'o' stands for 'outer domain' and Ω^o will play the role of the unperturbed outer domain in which we make two holes. To do so, we take two points

Please cite this article in press as: M. Dalla Riva, P. Musolino, The Dirichlet problem in a planar domain with two moderately close holes, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.04.006