ARTICLE IN PRESS YJDEQ:8796

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

Secondary bifurcation for a nonlocal Allen–Cahn equation ☆

Kousuke Kuto [a,*], Tatsuki Mori [b], Tohru Tsujikawa [c], Shoji Yotsutani [d]

[a] *Department of Communication Engineering and Informatics, The University of Electro-Communications, Chofu, Tokyo 182-8585, Japan*

[b] *Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka, 560-8531, Japan*

[c] *Faculty of Engineering, University of Miyazaki, Miyazaki, 889-2192, Japan*

[d] *Department of Applied Mathematics and Informatics, Ryukoku University, Seta, Otsu, 520-2194, Japan*

Received 16 February 2016; revised 30 January 2017

Abstract

This paper studies the Neumann problem of a nonlocal Allen–Cahn equation in an interval. A main result finds a symmetry breaking (secondary) bifurcation point on the bifurcation curve of solutions with odd-symmetry. Our proof is based on a level set analysis for the associated integral map. A method using the complete elliptic integrals proves the uniqueness of secondary bifurcation point. We also show some numerical simulations concerning the global bifurcation structure.

MSC: 34C23; 34B10; 37G40

Keywords: Allen–Cahn equation; Nonlocal term; Bifurcation; Symmetry breaking; Complete elliptic integrals

☆ K. Kuto was supported by Grant-in-Aid for Scientific Research (C) 15K04948. T. Tsujikawa was supported by Grant-in-Aid for Scientific Research (C) 17K05334. S. Yotsutani was supported by Grant-in-Aid for Scientific Research (C) 15K04972. This work was supported by the Joint Research Center for Science and Technology of Ryukoku University in 2016.

* Corresponding author.

E-mail addresses: k-kuto@uec.ac.jp (K. Kuto), tatsuki7837@gmail.com (T. Mori), tujikawa@cc.miyazaki-u.ac.jp (T. Tsujikawa), shoji@math.ryukoku.ac.jp (S. Yotsutani).

http://dx.doi.org/10.1016/j.jde.2017.04.010

Please cite this article in press as: K. Kuto et al., Secondary bifurcation for a nonlocal Allen–Cahn equation, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.04.010

1. Introduction

This paper is concerned with the Neumann problem for a nonlinear ordinary differential equation with nonlocal term:

$$\begin{cases} -du_{xx} = (1-u^2)\left(u - \dfrac{\mu}{2}\displaystyle\int_{-1}^{1} u\,\mathrm{d}x\right), & x \in I := (-1,1), \\ u_x(\pm 1) = 0, & \\ u_x(x) \geq 0, & x \in I, \end{cases} \tag{1.1}$$

where $d > 0$ and $\mu > 0$ are parameters. Such a class of Allen–Cahn type equations with nonlocal term appears in some reaction–diffusion models (e.g., [3,6,7,9,16]). Hence (1.1) has three constant solutions $u = 0$ and $u = \pm 1$. Since all solutions of (1.1) without the last condition can be constructed by connecting rescaled functions of monotone solutions, then we are restricted on nondecreasing solutions. Moreover, it is easily verified that if $u(x)$ is a solution of (1.1), then $-u(-x)$ is also a solution.

We note that (1.1) is equivalent to the Neumann problem of a perturbed Allen–Cahn equation

$$\begin{cases} -du_{xx} = (1-u^2)(u-\lambda), & x \in I, \\ u_x(\pm 1) = 0, & \\ u_x(x) \geq 0, & x \in I \end{cases} \tag{1.2}$$

with the nonlocal constraint

$$\lambda = \frac{\mu}{2}\int_{-1}^{1} u\,\mathrm{d}x. \tag{1.3}$$

Our interest is the following set of nonconstant solutions:

$$\mathcal{S} = \left\{ (u,d) \in C^2_\nu(\overline{I}) \times \mathbb{R}_+ \ :\ u \text{ is a nonconstant solution of (1.1)} \right\}, \tag{1.4}$$

where $C^2_\nu(\overline{I}) := \{u \in C^2(\overline{I}) \ :\ u'(\pm 1) = 0\}$ and $\mathbb{R}_+ := (0,\infty)$. Without the nonlocal term, (1.1) is reduced to the 1D stationary Allen–Cahn equation:

$$\begin{cases} -du_{xx} = (1-u^2)u, & x \in I, \\ u_x(\pm 1) = 0, & \\ u_x(x) \geq 0, & x \in I, \end{cases} \tag{1.5}$$

and the bifurcation structure of the set

$$\Gamma_0 = \left\{ (u,d) \in C^2_\nu(\overline{I}) \times \mathbb{R}_+ \ :\ u \text{ is a nonconstant solution of (1.5)} \right\}$$

is well known as follows (e.g., Chafee and Infante [2], Schaaf [13]):

Please cite this article in press as: K. Kuto et al., Secondary bifurcation for a nonlocal Allen–Cahn equation, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.04.010