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# The diffusive logistic model with a free boundary in a heterogeneous time-periodic environment ☆

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**Abstract**

This paper is concerned with a diffusive logistic model with advection and a free boundary in a spatially heterogeneous and time periodic environment. Such a model may be used to describe the spreading of a new or invasive species with the free boundary representing the expanding front. Under more general assumptions on the initial data and the function standing for the intrinsic growth rate of the species, sharp criteria for spreading and vanishing are established, and estimates for spreading speed when spreading occurs are also derived. The obtained results considerably improve and complement the existing ones, especially those of [11,25].



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*Keywords:* Diffusive logistic equation; Free boundary; Advection; Heterogeneous time-periodic environment; Spreading-vanishing dichotomy; Spreading speed

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## 1. Introduction

In this paper, we study the diffusive logistic equation with advection and a free boundary:

$$\begin{cases} u_t - du_{xx} - qu_x = u(\alpha(t,x) - \beta(t,x)u), & t>0,\ 0<x<h(t), \\ u_x(t,0)=0,\ u(t,h(t))=0, & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0)=h_0, u(0,x)=u_0(x), & 0\le x\le h_0. \end{cases} \tag{1.1}$$

Problem (1.1) may be used to describe the evolution of an invasive species in a heterogeneous time-periodic environment, in which $u(t,x)$ represents the population density of the single species at time $t$ and location $x$, $x = h(t)$, acting as the spreading front, is the free boundary to be determined, and the initial function $u_0(x)$ stands for the population density at its early stage of introduction. The coefficient functions $\alpha$ and $\beta$ can be interpreted, respectively, as the intrinsic growth rate of the species and its intra-specific competition, and the positive constant $d$ is the random diffusion rate and the nonnegative constant $q$ is the coefficient of the term $u_x$ which accounts for the influence of advection from 0 towards the moving front $h(t)$. A deduction of these conditions from ecological consideration can be found in [2].

Throughout this paper, we assume that $h_0$, $\mu$, $d$ are positive constants, and $u_0 \in \mathcal{H}(h_0)$ with

$$\mathcal{H}(h_0) := \Big\{\phi \in C([0,h_0]) : \phi'(0) = \phi(h_0) = 0,\ \phi(x) > 0 \text{ in } (0,h_0)\Big\}.$$

The functions $\alpha$ and $\beta$ satisfy the following conditions:

$$\begin{cases} \text{(i)} & \alpha,\ \beta \in C^{\nu_0/2,\nu_0}(\mathbb{R}\times[0,\infty)) \text{ for some } \nu_0 \in (0,1); \\ \text{(ii)} & \text{there are positive constants } \kappa_1,\ \kappa_2 \text{ such that} \\ & \alpha(t,x) \le \kappa_2,\quad \kappa_1 \le \beta(t,x) \le \kappa_2,\quad \forall x \in [0,\infty),\ t\in\mathbb{R}; \\ \text{(iii)} & \alpha(t,x),\ \beta(t,x) \text{ are } T\text{-periodic in } t \text{ for a fixed } T>0, \text{ that is,} \\ & \alpha(t,x) = \alpha(t+T,x),\ \beta(t,x) = \beta(t+T,x),\quad \forall x\in[0,\infty),\ t\in\mathbb{R}. \end{cases} \tag{1.2}$$

In what follows, let us briefly discuss the motivation of the present work by recalling some existing results on problem (1.1). When $\alpha$ and $\beta$ are positive constants, (1.1) with no advection term (i.e., $q = 0$) was first studied in [9] for the spreading of a new or invasive species. In such a case, it is proved that if

$$u_0 \in C^2([0,h_0]),\ u_0'(0) = u_0(h_0) = 0,\ u_0(x) > 0 \ \text{ in } (0,h_0),$$

(1.1) admits a unique solution $(u,h)$ with $u(t,x) > 0$ and $h'(t) > 0$ for all $t > 0$ and $0 \le x < h(t)$, and a spreading–vanishing dichotomy holds; namely, there is a spatial barrier $R^* > 0$ such that either

- **Spreading:** the free boundary crosses the barrier at some finite time (i.e., $h(t_0) \ge R^*$ for some $t_0 \ge 0$), and then goes to infinity as $t \to \infty$ (i.e., $\lim_{t\to\infty} h(t) = \infty$), and the population spreads to the entire space; or