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YJDEQ:8836

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Lyapunov type characterization of hyperbolic behavior ☆

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Received 28 January 2017

Abstract

We give a complete characterization of the uniform hyperbolicity and nonuniform hyperbolicity of a cocycle with values in the space of bounded linear operators acting on a Hilbert space in terms of the existence of appropriate quadratic forms. Our work unifies and extends many results in the literature by considering the general case of not necessarily invertible cocycles acting on a Hilbert space over an arbitrary invertible dynamics. As a nontrivial application of, we study the persistence of hyperbolicity under small linear perturbations.

MSC: primary 37D99

Keywords: Hyperbolic behavior; Lyapunov functions; Robustness

1. Introduction

Our main objective is to give a complete characterization of the uniform hyperbolicity and nonuniform hyperbolicity of a cocycle with values in the space of bounded linear operators acting on a Hilbert space in terms of the existence of appropriate quadratic forms. Our work unifies and

☆ L.B. and C.V. were supported by FCT/Portugal through UID/MAT/04459/2013. D.D. was supported in part by an Australian Research Council Discovery Project DP150100017 and Croatian Science Foundation under the project IP-2014-09-2285.

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http://dx.doi.org/10.1016/j.jde.2017.04.041

extends many results in the literature by considering the general case of not necessarily invertible cocycles acting on a Hilbert space over an arbitrary invertible dynamics $f: M \to M$.

For example, when $M = \mathbb{Z}$ and $f(n) = n + 1$, the concept of hyperbolicity introduced in Section 3 reduces to the notion of a uniform exponential dichotomy. This notion was essentially introduced by Perron in [17] and plays a central role in the qualitative theory of dynamical systems. We refer the reader to [6,9,10,21] for details and further references. In the same setting, the concept of nonuniform hyperbolicity considered in Section 5 includes the notion of a nonuniform exponential dichotomy as a particular case. We refer to [4] for the discussion of many related developments.

Many works in the literature have been devoted to the characterization of an exponential dichotomy in terms of the existence of appropriate quadratic forms. For some early contributions, we refer to the work of Maizel [14], Coppel [6,7] and Papaschinopoulos [16]. For more recent work dealing with nonuniform exponential dichotomies, see [5]. We emphasize that all these works consider only the particular case of an invertible finite-dimensional dynamics. Moreover, to the best of our understanding, the arguments in those works cannot be extended to our setting. This forced us to develop a different approach that relies on the spectral characterization of the hyperbolic behavior (see Section 2) and on the characterization of hyperbolic operators given in [8].

On the other hand, the generality of our setting enables to consider more complicated forms of hyperbolicity. For example, when f is differentiable, the notion of hyperbolicity considered in Section 3 reduces to the classical concept of uniform hyperbolicity introduced and studied by Smale [22] and Anosov [1]. The notion of uniform hyperbolicity was characterized in terms of the existence of quadratic forms by Lewowicz [12,13] but again only in the finite-dimensional setting and in the particular case of derivative cocycles. In the present work, we extend the results in [12] to an arbitrary noninvertible cocycle acting on an infinite-dimensional Hilbert space.

Moreover, the notion of hyperbolicity considered in Section 5 includes the concept of nonuniform hyperbolicity in the sense of Pesin [3,18] in the particular case of a finite-dimensional setting when the function K is tempered (see also [15,20] for related work in the infinite-dimensional setting). See [11] and the references therein for generalizations of the work of Lewowicz in this context. However, these works give characterizations of systems with nonzero Lyapunov exponents rather then of nonuniformly hyperbolic cocycles.

As a nontrivial application of our characterizations of the hyperbolic behavior, we study the persistence of hyperbolicity under small linear perturbations. This problem has a long history, especially in relation to exponential dichotomies. We refer the reader to [4,19] for details and further references.

2. Preliminaries

We first introduce some notions and results related to hyperbolicity that will be used throughout the paper.

2.1. Hyperbolic operators

Let $B(X)$ be the set of all bounded linear operators acting on a Hilbert space X. Given self-adjoint operators $A, B \in B(X)$, we write $A \leq B$ if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in X$. Moreover, an operator $A \in B(X)$ is said to be *hyperbolic* if its spectrum does not intersect the unit circle $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.