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Suitable weak solutions to the 3D Navier–Stokes equations are constructed with the Voigt approximation

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Abstract

In this paper we consider the Navier–Stokes equations supplemented with either the Dirichlet or vorticity-based Navier slip boundary conditions. We prove that weak solutions obtained as limits of solutions of the Navier–Stokes–Voigt model satisfy the local energy inequality, and we also prove certain regularity results for the pressure. Moreover, in the periodic setting we prove that if the parameters are chosen in an appropriate way, then we can construct suitable weak solutions through a Fourier–Galerkin finite-dimensional approximation in the space variables.

MSC: primary 35Q30; secondary 35A35, 76M20

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1. Introduction

We prove that weak solutions to the 3D Navier–Stokes Equations (1.1) (from now on NSE) obtained as limits of solutions to the Navier–Stokes–Voigt model (1.6) (from now on NSV) are *suitable weak solutions*.

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To set the problem, we recall that the initial boundary value problem for the incompressible NSE with unit viscosity, zero external force, and in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ is

$$
\begin{aligned}
u_t - \Delta u + (u \cdot \nabla) u + \nabla p &= 0 && \text{in } (0,T) \times \Omega, && (1.1)\\
\operatorname{div} u &= 0 && \text{in } (0,T) \times \Omega, && (1.2)\\
u &= 0 && \text{on } (0,T) \times \Gamma, && (1.3)\\
u(x,0) &= u_0(x) && \text{in } \Omega, && (1.4)
\end{aligned}
$$

where $u: (0,T) \times \Omega \to \mathbb{R}^3$ is the velocity vector field, $p: (0,T) \times \Omega \to \mathbb{R}$ is the scalar pressure, and $u_0(x)$ is a divergence-free vector initial datum. We are writing the problem with vanishing Dirichlet boundary conditions, but we will treat also a Navier-type boundary condition.

It is well-known that important open problems as global regularity and uniqueness of weak solutions for the 3D NSE are still open and very far from being solved, see Galdi [19] and Constantin and Foias [16]. A keystone regularity result for weak solutions is the partial regularity theorem of Caffarelli, Kohn, and Nirenberg [13] which asserts that the set of interior (possible) singularities has vanishing one-dimensional parabolic Hausdorff measure. For partial regularity results up-to-the-boundary, under Dirichlet boundary conditions, see [28]. In the case of Navier boundary condition (1.10) we could not find a specific reference, however we suspect and conjecture that similar results of partial regularity can be obtained also in this case with minor changes.

In any case, the partial regularity theorem holds for a particular subclass of Leray weak solutions, called starting from [13] *"suitable weak solutions,"* see Definition 2.1. Beside technical regularity properties, the most important additional requirement of suitable weak solutions (see Scheffer [36]) is the following inequality, often called in literature *local or generalized energy inequality*:

$$
\partial_t \left(\frac{1}{2}|u|^2\right) + \nabla \cdot \left(\left(\frac{1}{2}|u|^2 + p\right) u\right) - \Delta \left(\frac{1}{2}|u|^2\right) + |\nabla u|^2 \le 0, \tag{1.5}
$$

in $\mathcal{D}'((0,T) \times \Omega)$.

Since at present results of uniqueness for weak solutions are not known, we cannot exclude that each method used to construct weak solutions can produce its own class of solutions and these solutions could possibly not satisfy the local energy inequality. For this particular issue see also the recent review in Robinson, Rodrigo, and Sadowski [35]. It is then a relevant question to check whether weak solutions obtained by different methods are suitable or not, especially those constructed with methods which are well established by physical or computational motivations. Together with a construction by retarded mollifiers, the authors in [13] also recall that in [36] existence of suitable solution (even if the name did not exist yet) has been obtained just for the Cauchy problem without external force. The technical improvements to obtain partial regularity with external forces in the natural $L^2(\Omega)$ space – or even $H^{-1/2}(\Omega)$ – arrived only recently with the work of Kukavica [27] (in fact in [13] the force needed to be in $L^{\frac{5}{3}+\delta}(\Omega)$). Here we do not consider external forces and we are mainly treating the problem of showing that weak solutions constructed by certain approximations satisfy (1.5). This mainly relies on proving appropriate estimates on the pressure.

In the development of the concept of local energy inequality we recall – in earlier times – the two companion papers by Beirão da Veiga [2,3] dealing with the hyper-viscosity and a general