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Smoothing effects for the filtration equation with different powers

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Abstract

We study the nonlinear diffusion equation $u_t = \Delta\phi(u)$ on general Euclidean domains, with homogeneous Neumann boundary conditions. We assume that $\phi'(u)$ is bounded from below by $|u|^{m_1-1}$ for small $|u|$ and by $|u|^{m_2-1}$ for large $|u|$, the two exponents m_1, m_2 being possibly different and larger than one. The equality case corresponds to the well-known porous medium equation. We establish sharp short- and long-time L^{q_0}–L^∞ smoothing estimates: similar issues have widely been investigated in the literature in the last few years, but the Neumann problem with different powers had not been addressed yet. This work extends some previous results in many directions.

Keywords: Filtration equation; Porous medium equation; Neumann problem; Smoothing effects; Sobolev inequalities; Gagliardo–Nirenberg inequalities

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1. Introduction

The present paper is devoted to the study of *smoothing* and *asymptotic* properties for solutions of the following *filtration equation* with homogeneous *Neumann* boundary conditions:

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } \Omega \times \mathbb{R}^+ , \\ \frac{\partial \phi(u)}{\partial n} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ , \\ u(0) = u_0 & \text{in } \Omega , \end{cases} \tag{1.1}$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a (locally absolutely) continuous and strictly increasing function vanishing at zero, Ω is a general domain of $\mathbb{R}^N$ (not necessarily bounded or regular) and u_0 is an initial datum having suitable integrability properties that we shall specify below. Keeping in mind the widely studied case $\phi(u) = |u|^{m-1}u$ (let $m > 1$), we can also refer to (1.1) as *generalized porous medium equation*, in agreement with [51]. In fact we shall assume throughout, with the exception of Section 3, that ϕ is $C^1(\mathbb{R})$ and satisfies the following hypotheses:

$$\phi(0) = 0 , \tag{1.2}$$

$$c_1 \, |u|^{m_1-1} \le \phi'(u) \quad \forall u : \, |u| \in [0, 1] , \tag{1.3}$$

$$c_2 \, |u|^{m_2-1} \le \phi'(u) \quad \forall u : \, |u| > 1 , \tag{1.4}$$

for some exponents $m_1, m_2 > 1$ and positive constants c_1, c_2. In other words, if we think of (1.3)–(1.4) as equalities, we are allowing (1.1) to be like a porous medium equation with exponent m_1 where the solution is small and like a porous medium equation with another exponent m_2 where the solution is large. We shall see that m_2 is associated with short-time behaviour, whereas m_1 is associated with long-time asymptotics. It turns out that, to our purposes, the only requirements that count are bounds from *below* on ϕ' like (1.3)–(1.4), so that actually ϕ may significantly deviate from powers.

Recently, as concerns the straight porous-medium nonlinearity $\phi(u) = |u|^{m-1}u$, in [31, Theorem 3.2] it has been proved that, if Ω is bounded and regular and the spatial dimension N is greater than or equal to 3, the L^{q_0}–L^∞ smoothing effect

$$\|u(t)\|_\infty \le K \left(t^{-\frac{N}{2q_0+N(m-1)}} \, \|u_0\|_{q_0}^{\frac{2q_0}{2q_0+N(m-1)}} + \|u_0\|_{q_0} \right) \quad \forall t > 0 \tag{1.5}$$

holds for all $q_0 \in [1, \infty)$ and a suitable $K > 0$. As for long-time asymptotics, such estimate can be improved depending on whether $\overline{u}_0 = 0$ or $\overline{u}_0 \neq 0$, where $\overline{u}_0$ is the mean value of the initial datum. In the case $\overline{u}_0 = 0$, it is shown in [31, Theorem 4.1] that

$$\|u(t)\|_\infty \le K_1 \, t^{-\frac{N}{2q_0+N(m-1)}} \left(K_2 \, t + \|u_0\|_{q_0}^{1-m} \right)^{-\frac{2q_0}{(m-1)[2q_0+N(m-1)]}} \quad \forall t > 0 \tag{1.6}$$

holds for $q_0 \in [1, \infty)$ and suitable $K_1, K_2 > 0$, whereas in the case $\overline{u}_0 \neq 0$ [31, Theorem 4.3] establishes that

$$\|u(t) - \overline{u}_0\|_\infty \le G \, e^{-\frac{m|\overline{u}_0|^{m-1}}{C_P^2} \, t} \quad \forall t \ge 1 , \tag{1.7}$$