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YJDEQ:8813

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Journal of Differential Equations
www.elsevier.com/locate/jde

The well-posedness of the incompressible Magnetohydro Dynamic equations in the framework of Fourier–Herz space

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Received 16 December 2016

Abstract

In this paper, we study the well-posedness and the blow-up criterion of the mild solution for the 3D incompressible MHD equations in the framework of Fourier–Herz space involving highly oscillating function. First, we study the well-posedness of the incompressible MHD equations by establishing the smoothing effect in the mixed time–space Fourier–Herz space, which include the local in time for large initial data as well as the global well-posedness for small initial data. Next, we prove the blow-up criterion, that is, if $u \in \widetilde{L_T^{r_1}} F\dot{B}_{p,q}^{2-\frac{3}{p}+\frac{2}{r_1}}$ and $b \in \widetilde{L_T^{r_2}} F\dot{B}_{p,q}^{2-\frac{3}{p}+\frac{2}{r_2}}$ for $1 \le r_1, r_2 < \infty$, the mild solution to the MHD equations can be extended beyond $t = T$. More importantly, we give a better blow-up criterion in which we require velocity field $u(t) \in \widetilde{L_T^{r}} F\dot{B}_{p,q}^{2-\frac{3}{p}+\frac{2}{r}}$ only.

Keywords: Magnetohydro Dynamic equations; Well-posedness; Fourier–Herz space; The blow-up criterion

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http://dx.doi.org/10.1016/j.jde.2017.04.027
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1. Introduction

In this paper, we consider the Cauchy problem of the incompressible magneto hydrodynamics (MHD) equations in $\mathbb{R}^3 \times \mathbb{R}^+$:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = (b\cdot\nabla)b, \\ \partial_t b - \mu\Delta b + (u\cdot\nabla)b = (b\cdot\nabla)u, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ (u,b)|_{t=0} = (u_0, b_0). \end{cases} \tag{MHD}$$

Here, u, b and p describe the velocity field, the magnetic field and the pressure respectively. Besides, $\nu > 0$ is the kinematic viscosity and $\mu > 0$ is the magnetic diffusivity.

The incompressible MHD system coming from magneto-fluid mechanics is the study of the magnetic properties of electrically conducting fluids. Roughly speaking, it is a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. In the past years, many famous physicists and mathematicians study them and made more progress. In [16], Leary gave the concept of weak solutions and proved global existence of weak solutions for the initial data $u_0 \in L^2(\mathbb{R}^3)$ to the Cauchy problem for the 3D incompressible Navier–Stokes equation. Follow the study line of the incompressible Navier–Stokes equations, we see that the solutions of (MHD) also enjoy L^2 energy estimate:

$$\int_{\mathbb{R}^3} (|u(t)|^2 + |b(t)|^2)\,\mathrm{d}x + 2\nu \int_0^t \int_{\mathbb{R}^3} |\nabla\otimes u|^2\,\mathrm{d}x\mathrm{d}s + 2\mu \int_0^t \int_{\mathbb{R}^3} |\nabla\otimes b|^2\,\mathrm{d}x\mathrm{d}s$$
$$\leq \int_{\mathbb{R}^3} |u_0|^2 + |b_0|^2\,\mathrm{d}x.$$

Based on this energy estimate, Duvaut and Lions [9] and Sermange and Temam [22] proved that there exists at least one global-in-time L^2-weak solution (u,b) satisfying

$$(u,b) \in \big(L^\infty([0,T); L^2(\mathbb{R}^3))\big)^2 \cap \big(L^2([0,T); H^1(\mathbb{R}^3))\big)^2 \quad \text{for } \forall T > 0.$$

However, the uniqueness and regularity of weak solution for the 3D MHD equations have remained an open problem.

The another study line is to study the strong solutions of problem (MHD). For system (MHD), it is well-known that if (u,b) is the solution corresponding to the initial data (u_0,b_0), then $(u_\lambda, b_\lambda) \triangleq (\lambda u(\lambda^2 t, \lambda(x)), \lambda b(\lambda^2 t, \lambda(x)))$ is also a solution with initial data $(u_{0\lambda}, b_{0\lambda}) \triangleq \lambda(u_0(\lambda x), b_{0\lambda}(\lambda x))$. Thus, one can define the critical space which is invariant under the above scaling. As $b = 0$, it reduces to the 3D incompressible Navier–Stokes equations. Inspired by this scaling analysis, Fujita and Kato [10] showed that it is locally well-posed in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and globally well-posed in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for the small initial data. Later on, Kato [14] obtained the similar result for the initial data in $L^3(\mathbb{R}^3)$. After that, Cannone [3] and Planchon [20] give the similar result in Besov spaces $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ with $1 \leq p < \infty$. Based on it, Cannone, Miao, Prioux and Yuan [5] proved that, for any $(u_0,b_0) \in \dot{B}^{\frac{3}{p}-1}_{p,q}(\mathbb{R}^3)$, there exists a positive time T and a unique local-in-time mild solution (u,b) to (MHD) such that

Please cite this article in press as: J. Li, X. Zheng, The well-posedness of the incompressible Magnetohydro Dynamic equations in the framework of Fourier–Herz space, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.04.027