Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 263 (2017) 3533–3549

Journal of Differential Equations

www.elsevier.com/locate/jde

Non-autonomous maximal regularity for forms given by elliptic operators of bounded variation ☆

Stephan Fackler

Institute of Applied Analysis, Ulm University, Helmholtzstr. 18, 89069 Ulm, Germany

Received 15 November 2016; revised 7 April 2017
Available online 18 May 2017

Abstract

We show maximal L^p-regularity for non-autonomous Cauchy problems provided the trace spaces are stable in some parameterized sense and the time dependence is of bounded variation. In particular on $L^2(\Omega)$, for Lipschitz domains Ω and under mixed boundary conditions, we obtain maximal L^p-regularity for all $p \in (1, 2]$ for elliptic operators with coefficients $a_{ij} \colon \Omega \to \mathbb{C}$ satisfying $a_{ij}(\cdot, x) \in \mathrm{BV}$ uniformly in $x \in \Omega$.

MSC: primary 35B65; secondary 47A07

1. Introduction

Let X be a Banach space and $(A(t))_{t\in[0,T)}$ closed operators on X. For an inhomogeneity $f \colon [0, T) \to X$ and an initial value $u_0 \in X$ we consider the non-autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) + A(t)u(t) = f(t) \\ u(0) = u_0. \end{cases} \tag{NACP}$$

For $p \in (1, \infty)$ we say that (NACP) has *maximal L^p-regularity* for $u_0 \in X$ if for all $f \in L^p([0, T); X)$ there exists a unique solution in $\mathrm{MR}_p^A([0, T))$, the space of all measurable

☆ This work was supported by the DFG grant AR 134/4-1 “Regularität evolutionärer Probleme mittels Harmonischer Analyse und Operatortheorie”. The author thanks M. Egert for sharing his insights on the Kato square root problem and the anonymous referee for his careful reading.

E-mail address: stephan.fackler@uni-ulm.de.

http://dx.doi.org/10.1016/j.jde.2017.04.036
0022-0396/

functions $u\colon [0,T) \to X$ with $\dot{u} \in L^p([0,T);X)$, $u(t) \in D(A(t))$ for almost all $t \in [0,T)$ and $A(\cdot)u(\cdot) \in L^p([0,T);X)$. Of particular interest for applications is the case where $A(t)$ are elliptic operators with boundary conditions. Maximal regularity can then be used in various ways to show existence of solutions for quasilinear equations and to study the asymptotic behaviour of their solutions [33].

In the autonomous case $A(t) = A$ maximal regularity is well understood. A closed operator $A\colon D(A) \to X$ has maximal L^p-regularity for $u_0 \in X$ if and only if A is $\mathcal{R}$-sectorial and u_0 lies in the real interpolation space $\operatorname{Tr}_p A := (D(A), X)_{1/p,p}$. The notation $\operatorname{Tr}_p A$ is justified by the fact that in the autonomous case one has $\mathrm{MR}_p^A([0,T]) \hookrightarrow C([0,T]; \operatorname{Tr}_p A)$ and, further, that for every $x \in \operatorname{Tr}_p A$ one can find $u \in \mathrm{MR}_p^A([0,T])$ with $u(0) = x$ [3, Theorem III.4.10.2], i.e. the interpolation space is identical with the trace space.

The simplest situation occurs when the operators $A(t)$ are associated with a non-autonomous form on some Hilbert space. For that, we consider a Hilbert space V densely embedded into a second Hilbert space H. This induces the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$. Further, a non-autonomous bounded coercive sesquilinear form $a\colon [0,T] \times V \times V \to \mathbb{C}$ is given, i.e. $a(t,\cdot,\cdot)$ is sesquilinear for all $t \in [0,T]$ and satisfies for some $\alpha, M > 0$ and all $u, v \in V$

$$\begin{aligned} |a(t,u,v)| &\le M \left\|u\right\|_V \left\|v\right\|_V, \\ \operatorname{Re} a(t,u,u) &\ge \alpha \left\|u\right\|_V^2. \end{aligned} \tag{A}$$

For fixed $t \in [0,T]$ the sesquilinear form $a(t,\cdot,\cdot)$ induces a bounded operator $\mathcal{A}(t)\colon V \to V'$, which is also an unbounded operator on V'. We denote its part in H by $A(t)$. By a classical result of Lions [15, p. 513, Theorem 2], the problem (NACP) for $\mathcal{A}$ satisfies maximal L^2-regularity if $t \mapsto a(t,u,v)$ is measurable for all $u, v \in V$. However, maximal L^2-regularity for the operator $A(t)$ on H is far more involved. For some time there was the hope that maximal L^2-regularity for A holds for all $u_0 \in \operatorname{Tr}_2 A(0)$ and measurable non-autonomous forms. Requiring additionally the symmetry of a, i.e. $a(t,u,v) = \overline{a(t,v,u)}$ for all $t \in [0,T]$ and $u, v \in V$, this problem was explicitly asked by Lions for $u_0 = 0$ [28, p. 68]. Dier observed that maximal L^2-regularity can fail due to a single jump of the form, see [12, Section 5.2] and [1, Example 5.1]. He used the failure of the Kato square root property for the construction of his counterexamples. Later, the author gave Hölder continuous and symmetric counterexamples to Lions' original question [18].

In the positive direction Dier showed maximal L^2-regularity for *symmetric forms* of bounded variation [13]. Note that this result allows jumps and therefore does not hold for general forms. However, apart from self-adjoint operators *every* elliptic operator in divergence form has the Kato square root property by the celebrated result [7]. In view of applications it is therefore interesting to ask for maximal regularity of non-symmetric elliptic operators.

Assuming BV-coefficients, in one of our main results we show that maximal L^2-regularity indeed holds for *non-symmetric* elliptic operators and also for systems of elliptic operators under a broad range of domains and boundary conditions including mixed ones. Our methods are not restricted to the case $p = 2$. Remarkably, we establish maximal L^p-regularity for $p \in (1,2)$ for arbitrary forms of bounded variation *independently* of the Kato square root property. In Section 7 we construct a non-autonomous elliptic operator of bounded variation that does not have maximal L^p-regularity for any $p > 2$. Hence, the aforementioned results are optimal and only the critical case $p = 2$ requires the Kato square root property.