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Schrödinger operators involving singular potentials and measure data

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Abstract

We study the existence of solutions of the Dirichlet problem for the Schrödinger operator with measure data

$$\begin{cases} -\Delta u + Vu = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We characterize the finite measures μ for which this problem has a solution for every nonnegative potential V in the Lebesgue space $L^p(\Omega)$ with $1 \le p \le \frac{N}{2}$. The full answer can be expressed in terms of the $W^{2,p}$ capacity for $p > 1$, and the $W^{1,2}$ (or Newtonian) capacity for $p = 1$. We then prove the existence of a solution of the problem above when V belongs to the real Hardy space $H^1(\Omega)$ and μ is diffuse with respect to the $W^{2,1}$ capacity.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded and smooth set in dimension $N \geq 2$, and let $V \in L^1(\Omega)$ be a nonnegative function. We address in this paper the question of existence of solutions of the linear Dirichlet problem with measure data

$$\begin{cases} -\Delta u + Vu = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The variational solution of this problem when $\mu \in L^2(\Omega)$ can be obtained by a straightforward minimization of the associated energy functional

$$E(v) = \frac{1}{2} \int_\Omega \left(|\nabla v|^2 + V v^2 \right) - \int_\Omega \mu v,$$

which is bounded from below in $W_0^{1,2}(\Omega)$ since V is nonnegative. Using as a test function in (1.1) a suitable approximation of $\operatorname{sgn} u$, one deduces the absorption estimate

$$\|Vu\|_{L^1(\Omega)} \leq \|\mu\|_{L^1(\Omega)}. \tag{1.2}$$

When $\mu \in L^1(\Omega)$, the functional E need not be bounded from below, but one can use an approximation argument with L^2 functions to find a solution of (1.1), based on the linearity of the equation and the absorption estimate above; see [11,19]. In this case, the solution $u \in L^1(\Omega)$ satisfies $Vu \in L^1(\Omega)$ and the functional identity:

$$\int_\Omega u(-\Delta\zeta + V\zeta) = \int_\Omega \zeta\mu,$$

for every $\zeta \in C^\infty(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. In the sequel, we denote by $C_0^\infty(\overline{\Omega})$ the space of such test functions ζ. The right-hand side of this identity is well-defined even if μ is merely a finite Borel measure; in this case the integral is interpreted as integration of ζ with respect to μ. This is the notion of weak solution of the Dirichlet problem (1.1) which has been introduced by Littman, Stampacchia and Weinberger [22, Definition 5.1].

In contrast with the L^1 case, the existence of solutions of the Dirichlet problem (1.1) with measure data is more subtle. For example, in dimension $N \geq 3$, the equation

$$-\Delta u + \frac{u}{|x|^\alpha} = \delta_0 \quad \text{in } B_1,$$

where B_1 is the unit ball in $\mathbb{R}^N$ centered at 0, has no solution in the sense of distributions when $\alpha \geq 2$. Heuristically, $u(x)$ behaves like $\frac{1}{|x|^{N-2}}$, as the fundamental solution of the Laplacian in a neighborhood of 0, and this is incompatible with the requirement that $\frac{u}{|x|^\alpha} \in L^1_{\text{loc}}(B_1)$; see Proposition 9.1 below in the spirit of [6, Remark A.4]. On the contrary, for $\alpha < 2$, a solution does exist, and more generally Stampacchia [32, Théorème 9.1] proved that, for every nonnegative function $V \in L^p(\Omega)$ with $p > \frac{N}{2}$, the Dirichlet problem (1.1) has a solution for any finite measure μ.