ARTICLE IN PRESS YJDEQ:8542

Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

An application of the Nash–Moser theorem to the vacuum boundary problem of gaseous stars

Tetu Makino [1]

Received 7 April 2016; revised 17 July 2016

Abstract

We have been studying spherically symmetric motions of gaseous stars with physical vacuum boundary governed either by the Euler–Poisson equations in the non-relativistic theory or by the Einstein–Euler equations in the relativistic theory. The problems are to construct solutions whose first approximations are small time-periodic solutions to the linearized problem at an equilibrium and to construct solutions to the Cauchy problem near an equilibrium. These problems can be solved when $1/(\gamma - 1)$ is an integer, where γ is the adiabatic exponent of the gas near the vacuum, by the formulation by R. Hamilton of the Nash–Moser theorem. We discuss on an application of the formulation by J.T. Schwartz of the Nash–Moser theorem to the case in which $1/(\gamma - 1)$ is not an integer but sufficiently large.

MSC: 35L70; 35Q31; 35Q85; 76L10; 83C05

Keywords: Non-linear hyperbolic equations; Nash–Moser theory; Vacuum boundary; Spherically symmetric solutions; Gaseous stars

1. Introduction

In the previous works [13,14], we investigated the time evolution of spherically symmetric gaseous stars, either in the non-relativistic case governed by the Euler–Poisson equations [13], or in the relativistic case governed by the Einstein–Euler equations [14]. Our studies suppose that

E-mail address: makino@yamaguchi-u.ac.jp.

[1] Professor Emeritus at Yamaguchi University, Japan.

http://dx.doi.org/10.1016/j.jde.2016.09.042

the gas remains to be barotropic during the evolutions. That is, the pressure P is a given fixed function of the density ρ. We were assuming

(**A**): *P is a given smooth function of $\rho > 0$ such that $P > 0, dP/d\rho > 0$ for $\rho > 0$ and there are positive constants A, γ such that $1 < \gamma < 2$ and an analytic function Ω on a neighborhood of 0 such that $\Omega(0) = 1$ and*

$$P = A\rho^{\gamma}\Omega(\rho^{\gamma-1})$$

for $0 < \rho \ll 1$.

Assuming that there is an equilibrium with a finite radius at which the gas touches the vacuum, we investigated time-dependent solutions near this equilibrium. The existence of solutions whose first approximations are small time-periodic solutions to the linearized problem at the equilibrium, and the existence of solutions to the Cauchy problem near this equilibrium were established by applying the Nash–Moser theorem formulated by R. Hamilton, [4], which we shall call the **'Nash–Moser(–Hamilton) theorem'**.

But in order to apply this Nash–Moser(–Hamilton) theorem, we had to put the assumption

(**B**): *N is an even integer.*

Here the parameter N is determined from the approximate adiabatic exponent γ near the vacuum by

$$\frac{N}{2} = 1 + \frac{1}{\gamma - 1}, \qquad \text{or} \qquad \gamma = 1 + \frac{2}{N-2}.$$

Under the assumption (**B**), the function $(1 - x)^{N/2}$ is analytic at $x = 1$, and the smoothness of this function plays an essential rôle for the application of the Nash–Moser(–Hamilton) theorem.

However in many physically important cases, in which, e.g., $\gamma = 5/3, 7/5$ and so on, $N/2$ is not an integer. Therefore the open problem to apply the Nash–Moser theorem for the case in which N is not an even integer is very important.

The present study is a partial answer to this open problem. In fact, when N is very large, that is, $\gamma - 1$ is very small, the Nash–Moser theorem formulated by J.T. Schwartz, [16], which we shall call the **'Nash–Moser(–Schwartz) theorem'**, can be applied. To show this is the aim of this article.

The reason why we apply the Nash–Moser theorem is that we have to treat the so called 'physical vacuum boundary', that is, a boundary $\partial\Omega$ of the domain Ω on which $\rho > 0$ and outside which $\rho = 0$ such that

$$0 < -\frac{\partial}{\partial \mathbf{N}}\Big(\frac{dP}{d\rho}\Big) < +\infty,$$

where $\mathbf{N}$ is the outer normal vector of $\partial\Omega$. (Cf. [3,5].) Because of this singularity at the physical vacuum boundary, the nonlinear hyperbolic evolution equation to be considered involves a loss of the derivative regularities by which the usual iteration does not work. (Cf. [12, p. 49].) This difficulty has already been attacked by several scholars: D. Coutand and S. Shkoller, [2,3],