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YJDEQ:8543

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Comparative index and Sturmian theory for linear Hamiltonian systems ☆

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Received 27 July 2016; revised 27 September 2016

Abstract

The comparative index was introduced by J. Elyseeva (2007) as an efficient tool in matrix analysis, which has fundamental applications in the discrete oscillation theory. In this paper we implement the comparative index into the theory of continuous time linear Hamiltonian systems, study its properties, and apply it to obtain new Sturmian separation theorems as well as new and optimal estimates for left and right proper focal points of conjoined bases of these systems on bounded intervals. We derive our results for general possibly abnormal (or uncontrollable) linear Hamiltonian systems. The results turn out to be new even in the case of completely controllable systems. We also provide several examples, which illustrate our new theory.

MSC: 34C10

Keywords: Linear Hamiltonian system; Sturmian separation theorem; Proper focal point; Comparative index; Conjoined basis; Controllability

1. Introduction

In this paper we study oscillation properties of solutions of the linear Hamiltonian system

☆ This research was supported by the Czech Science Foundation under grant GA16-00611S.
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http://dx.doi.org/10.1016/j.jde.2016.09.043
0022-0396/

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \qquad t \in [a,b], \tag{H}$$

where $A, B, C : [a,b] \to \mathbb{R}^{n\times n}$ are given piecewise continuous matrix-valued functions on the interval $[a,b]$ such that $B(t)$ and $C(t)$ are symmetric and the Legendre condition holds, i.e.,

$$B(t) \geq 0 \quad \text{for all } t \in [a,b]. \tag{1.1}$$

Here $n \in \mathbb{N}$ is a given dimension and $a, b \in \mathbb{R}$, $a < b$, are fixed numbers. The main results of this paper are concerned with the Sturmian type separation theorems about the number of focal points of conjoined bases of (H) in the given interval. We present a novel approach to this problem, which is based on the so-called comparative index of two conjoined bases of (H), see (2.3) in Section 2 below.

System (H) is traditionally studied under the complete controllability assumption. This means that the only solution (x,u) of (H) with $x(t) \equiv 0$ on a subinterval of $[a,b]$ with positive length is the trivial solution $(x,u) \equiv 0$ on $[a,b]$, see e.g. [5,17,22,26,27]. In this case $t_0 \in [a,b]$ is a *focal point* of a conjoined basis (X,U) of (H) if $X(t_0)$ is singular, and then

$$m(t_0) := \operatorname{def} X(t_0) = \dim \operatorname{Ker} X(t_0)$$

is its multiplicity. We refer to Section 3 for the definition of a conjoined basis. Every conjoined basis of (H) then has finitely many focal points in $[a,b]$, and the numbers of focal points in $(a,b]$ or in $[a,b)$ of any two conjoined bases of (H) differ by at most n, see [22, Theorem 4.1.3, p. 126] and [26, Corollary 1, p. 366]. In addition, for one conjoined basis (X,U) of (H) the difference between the numbers of its focal points in $(a,b]$ and in $[a,b)$ equals the value

$$\operatorname{def} X(b) - \operatorname{def} X(a) = \operatorname{rank} X(a) - \operatorname{rank} X(b). \tag{1.2}$$

When the controllability assumption is absent, Kratz showed in [23, Theorem 3] the following crucial result.

Proposition 1.1. *Assume that (1.1) holds. Then for any conjoined basis (X,U) of (H) the kernel of $X(t)$ is piecewise constant on $[a,b]$, i.e., there is a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that* $\operatorname{Ker} X(t)$ *is constant on the open interval (t_j, t_{j+1}) for all $j \in \{0,1,\dots,m-1\}$ and*

$$\operatorname{Ker} X(t_j^-) \subseteq \operatorname{Ker} X(t_j), \qquad j \in \{1,2,\dots,m\}, \tag{1.3}$$

$$\operatorname{Ker} X(t_j^+) \subseteq \operatorname{Ker} X(t_j), \qquad j \in \{0,1,\dots,m-1\}. \tag{1.4}$$

The quantity $\operatorname{Ker} X(t_j^{\pm})$ denotes the limit of the constant set $\operatorname{Ker} X(t)$ as $t \to t_j^{\pm}$. The inclusions in (1.3) and (1.4) follow from the continuity of $X(t)$ on $[a,b]$. In the subsequent work [36], Wahrheit defined the point $t_0 \in (a,b]$ to be a *left proper focal point* of (X,U) if $\operatorname{Ker} X(t_0^-) \subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity

$$m_L(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^-) = \operatorname{rank} X(t_0^-) - \operatorname{rank} X(t_0). \tag{1.5}$$

In a similar way we define $t_0 \in [a,b)$ to be a right proper focal point of (X,U) by the condition $\operatorname{Ker} X(t_0^+) \subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity