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An avoiding cones condition for the Poincaré–Birkhoff Theorem

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Abstract

We provide a geometric assumption which unifies and generalizes the conditions proposed in [11,12], so to obtain a higher dimensional version of the Poincaré–Birkhoff fixed point Theorem for Poincaré maps of Hamiltonian systems.

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Keywords: Poincaré–Birkhoff Theorem; Avoiding cones condition; Hamiltonian systems; Periodic solutions

1. Introduction and main result

The seminal work of Henri Poincaré [21] gave rise to a huge amount of research, with the aim of better understanding the far-reaching consequences of the so-called *Poincaré's last geometric Theorem* or *Poincaré–Birkhoff Theorem*. Since then, however, a genuine generalization to higher dimensions of this planar fixed point theorem has never been found. We refer to [1,16] for a classical introduction, and to [8,18] for recent reviews on this topic. Recently, however, the first author and Antonio J. Ureña proposed in [11,12] a higher dimensional version of the

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Poincaré–Birkhoff Theorem which applies to Poincaré maps of Hamiltonian systems. The aim of this paper is to unify and generalize the geometrical conditions proposed there.

We consider the Hamiltonian system

$$\dot{z} = J\nabla H(t,z)\,, \tag{HS}$$

where $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ denotes the standard $2N \times 2N$ symplectic matrix, and we assume the Hamiltonian function $H\colon \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ to be $\mathcal{C}^\infty$-smooth, and T-periodic in its first variable t. (Actually, such a regularity assumption can be considerably weakened, as will be discussed below.) We denote by $\nabla H(t,z)$ the gradient with respect to the variable z.

For every $\zeta \in \mathbb{R}^{2N}$, we denote by $\mathcal{Z}(\cdot,\zeta)$ the unique solution of (HS) satisfying $\mathcal{Z}(0,\zeta) = \zeta$. We assume that these solutions can be continued to the whole time interval $[0,T]$, so that the Poincaré map $\mathcal{P}\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is well defined, by setting

$$\mathcal{P}(\zeta) = \mathcal{Z}(T,\zeta)\,,$$

and it is a diffeomorphism. The fixed points of $\mathcal{P}$ are associated with the T-periodic solutions of (HS).

For $z \in \mathbb{R}^{2N}$, we use the notation $z = (x,y)$, with $x = (x_1,\dots,x_N) \in \mathbb{R}^N$ and $y = (y_1,\dots,y_N) \in \mathbb{R}^N$, and we assume that $H(t,x,y)$ is 2π-periodic in each of the variables $x_1,\dots,x_N$. Under this setting, T-periodic solutions of (HS) appear in equivalence classes made of those solutions whose components $x_i(t)$ differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class. The same will be said for two fixed points of $\mathcal{P}$.

We now describe our geometrical setting, by introducing a family of closed cones associated to a particularly structured vector field.

Let $F\colon \mathbb{R}^N \to \mathbb{R}^N$ be a $\mathcal{C}^\infty$-smooth gradient function, i.e., there is a function $h\colon \mathbb{R}^N \to \mathbb{R}$ such that $F = \nabla h$. We define, for every $y \in \mathbb{R}^N$, the set $\mathcal{A}_F(y)$ as follows: a vector $v \in \mathbb{R}^N$ belongs to $\mathcal{A}_F(y)$ if and only if there exist a sequence $(y_n)_n$ of points in $\mathbb{R}^N$ and a sequence $(\mu_n)_n$ of non-negative real numbers such that

$$y_n \to y\,, \quad \text{and} \quad \mu_n F(y_n) \to v\,.$$

It can be easily seen that $\mathcal{A}_F(y)$ is a closed cone in $\mathbb{R}^N$.

Our main result is the following.

Theorem 1. *Let $F = \nabla h\colon \mathbb{R}^N \to \mathbb{R}^N$ be a $\mathcal{C}^\infty$-smooth function for which there are two constants $K > 0$ and $C > 0$ and a regular symmetric $N \times N$ matrix $\mathbb{S}$ such that*

$$|F(y) - \mathbb{S}y| \le C\,, \quad \text{when } |y| \ge K\,, \tag{1}$$

and set $D := F^{-1}(0)$. Writing

$$\mathcal{P}(x,y) = (x + \vartheta(x,y), \rho(x,y))\,, \tag{2}$$

suppose that

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