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Regularity of weak solutions of elliptic and parabolic equations with some critical or supercritical potentials

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Abstract

We prove Hölder continuity of weak solutions of the uniformly elliptic and parabolic equations

$$\partial_i (a_{ij}(x)\partial_j u(x)) - \frac{A}{|x|^{2+\beta}} u(x) = 0 \quad (A > 0, \quad \beta \geq 0), \tag{0.1}$$

$$\partial_i (a_{ij}(x,t)\partial_j u(x,t)) - \frac{A}{|x|^{2+\beta}} u(x,t) - \partial_t u(x,t) = 0 \quad (A > 0, \quad \beta \geq 0), \tag{0.2}$$

with critical or supercritical 0-order term coefficients which are beyond De Giorgi–Nash–Moser's Theory. We also prove, in some special cases, weak solutions are even differentiable.

Previously P. Baras and J. A. Goldstein [3] treated the case when $A < 0$, $(a_{ij}) = I$ and $\beta = 0$ for which they show that there does not exist any regular positive solution or singular positive solutions, depending on the size of $|A|$. When $A > 0$, $\beta = 0$ and $(a_{ij}) = I$, P. D. Milman and Y. A. Semenov [7,8] obtain bounds for the heat kernel.

MSC: 35D30; 35J15; 35K10

Keywords: Weak solutions; Elliptic; Parabolic; Hölder continuity; Critical; Supercritical potential

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1. Introduction

In this paper, we consider regularity of weak solutions of divergence form elliptic equations

$$\partial_i(a_{ij}\partial_j u) - \frac{A}{|x|^{2+\beta}}u = 0 \tag{1.1}$$

and parabolic equations

$$\partial_i(a_{ij}\partial_j u) - \frac{A}{|x|^{2+\beta}}u - \partial_t u = 0 \tag{1.2}$$

in the unit ball $B := B(0,1)$ (or $B \times \mathbb{R}_+$) in $\mathbb{R}^d$, with $d \geq 3$, $A > 0$, $\beta \geq 0$. Here $a_{ij} \in L^\infty(B)$ (or $L^\infty(\mathbb{R}_+, L^\infty(B))$), and the second order coefficient matrix $\left(a_{ij}\right)_{1\leq i,j\leq d}$ satisfies the uniformly elliptic condition:

$$\lambda I \leq \left(a_{ij}\right)_{1\leq i,j\leq d} \leq \Lambda I, \text{ for some } 0 < \lambda \leq \Lambda < \infty. \tag{1.3}$$

Here and below, we use the Einstein summation convention. We say $u \in H^1(B)$ is a weak solution of the elliptic equation of (1.1), if $\forall\ \psi \in C_0^\infty(B)$, there holds

$$\int_B a_{ij}(x)\partial_i\psi(x)\partial_j u(x)dx + \int_B \frac{A}{|x|^{2+\beta}}u(x)\psi(x)dx = 0, \tag{1.4}$$

where ∂_i indicates ∂_{x_i} here and below. Similarly, for the parabolic equation (1.2), we say $u \in L^2([0,T], H^1(B))$ is a weak solution, if $\forall\ \psi \in C_0^\infty(B \times [-T,T])$, there holds

$$\begin{aligned}
&-\int_0^T\int_B u(x,t)\partial_t\psi(x,t)dx + \int_0^T\int_B a_{ij}(x,t)\partial_i\psi(x,t)\partial_j u(x,t)dxdt \\
&+\int_0^T\int_B \frac{A}{|x|^{2+\beta}}u(x,t)\psi(x,t)dxdt = \int_B \psi(x,0)u(x,0)dx.
\end{aligned} \tag{1.5}$$

In the middle of the last century, De Giorgi [5], Nash [11] and Moser [9,10] developed new methods on the studying of elliptic and parabolic equation, which opened a new area on the study of regularity of weak solutions of elliptic and parabolic equations in divergence form:

$$\partial_i(a^{ij}\partial_j u) + b^i\partial_i u + cu = f + \partial_i f^i, \tag{1.6}$$

$$\partial_i(a^{ij}\partial_j u) + b^i\partial_i u + cu - \partial_t u = f + \partial_i f^i. \tag{1.7}$$

They proved that, under certain integrable conditions of the coefficients b^i, c and non-homogeneous terms f and f^i, weak solutions of equation (1.6), (1.7) have C^α Hölder continuity. A key condition for their theory for elliptic equations is that the coefficient of the 0-order term c

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