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Representation of solutions and large-time behavior for fully nonlocal diffusion equations

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Abstract

We study the Cauchy problem for a nonlocal heat equation, which is of fractional order both in space and time. We prove four main theorems:

(i) a representation formula for classical solutions,
(ii) a quantitative decay rate at which the solution tends to the fundamental solution,
(iii) optimal L^2-decay of mild solutions in all dimensions,
(iv) L^2-decay of weak solutions via energy methods.

The first result relies on a delicate analysis of the definition of classical solutions. After proving the representation formula we carefully analyze the integral representation to obtain the quantitative decay rates of (ii).

Next we use Fourier analysis techniques to obtain the optimal decay rate for mild solutions. Here we encounter the *critical dimension phenomenon* where the decay rate attains the decay rate of that in a bounded domain for large enough dimensions. Consequently, the decay rate does not anymore improve when the dimension increases. The theory is markedly different from that of the standard caloric functions and this substantially complicates the analysis.

Finally, we use energy estimates and a comparison principle to prove a quantitative decay rate for weak solutions defined via a variational formulation. Our main idea is to show that the L^2-norm is actually a

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subsolution to a purely time-fractional problem which allows us to use the known theory to obtain the result.

MSC: primary 35R11; secondary 45K05, 35C15, 47G20

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1. Introduction

We study the Cauchy problem for the diffusion equation

$$\partial_t^\alpha (u(t,x) - u_0) + \mathcal{L}u(t,x) = f(t,x) \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^d, \;\; 0 < \alpha \le 1, \tag{1.1}$$

where $u_0(x) = u(0,x)$ is the initial condition, ∂_t^α denotes the Riemann–Liouville fractional derivative if $\alpha \in (0,1)$ and $\mathcal{L}$ is a nonlocal elliptic operator of order $\beta \in (0,2]$. A standard example is the fractional Laplacian $\mathcal{L} = (-\Delta)^{\frac{\beta}{2}}$. The equation is nonlocal both in space and time and we call such a parabolic equation *a fully nonlocal diffusion equation*.

Our emphasis is on the decay properties, and for the space-fractional heat diffusion such questions have been studied, for instance, by Chasseigne, Chaves and Rossi in [13] as well as by Ignat and Rossi in [28]. For a more comprehensive account of the asymptotic theory in case $\alpha = 1$, we refer to [38]. The decay of solutions and behavior of the Barenblatt solution for the space-fractional porous medium equation have, in turn, been studied by Vazquez in [43]. In the present paper, we extend these developments – concerning the fundamental solutions, representation formulas and decay properties – to the above fully nonlocal equation. For the case $\beta = 2$, see earlier works by Vergara and Zacher in [44] and by Vergara and the present authors in [29]. For the regularity theory of nonlocal equations in case $\alpha = 1$ or $\beta = 2$, we refer to [11,7,22,4,5,31,2,50,49] and the references therein. For the elliptic theory, see for instance [8–10].

Nonlocal PDE models arise directly, and naturally, from applications. Time fractional diffusion equations are closely related to a class of Montroll–Weiss continuous time random walk (CTRW) models and have become one of the standard physics approaches to model *anomalous diffusion* processes [17,15,27,34]. For a detailed derivation of these equations from physics principles and for further applications of such models we refer to the expository review article of Metzler and Klafter in [35]. The fractional Laplacian arises in the modelling of jump processes and also in quantitative finance as a model for pricing American options [16,41]. The fully nonlocal diffusion equation, in particular, has been used in diffusion models, for instance, in [12] and [15].

Despite their importance for applications, the mathematical study of fully nonlocal diffusion problems of type (1.1) is relatively young. In a very recent paper Allen, Caffarelli and Vasseur [1] have studied the regularity of weak solutions to such problems. Even more recently, simultaneously to our work, Kim and Lim [32] have considered the behavior of fundamental solutions, whereas Cheng, Li and Yamamoto [14] have studied other aspects of the asymptotic theory. Apart from these papers, the study of the parabolic problem has mostly concentrated on the aforementioned cases $\alpha = 1$ or $\beta = 2$.

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