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J. Differential Equations 263 (2017) 202–239

Journal of Differential Equations

www.elsevier.com/locate/jde

Global existence and regularity of solutions for active liquid crystals

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Received 20 January 2017
Available online 3 March 2017

Abstract

We study the hydrodynamics of active liquid crystals in the Beris–Edwards hydrodynamic framework with the Landau–de Gennes Q-tensor order parameter to describe liquid crystalline ordering. The existence of global weak solutions in two and three spatial dimensions is established. In the two-dimensional case, by the Littlewood–Paley decomposition, the higher regularity of the weak solutions and the weak-strong uniqueness are also obtained.
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MSC: 35Q35; 76D05; 76A15

Keywords: Navier–Stokes equations; Active liquid crystals; Global well-posedness; Weak solutions; Strong solutions; Weak-strong uniqueness

1. Introduction

Liquid crystals are classical examples of mesophases that are intermediate between solids and liquids (*cf.* [10]). They often combine physical properties of both liquids and solids, and in general liquid crystals can be divided into thermotropic, lyotropic, and metallotropic phases,

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http://dx.doi.org/10.1016/j.jde.2017.02.035
0022-0396/Published by Elsevier Inc.

according to their different optical properties. Nematic liquid crystals are one of the most common liquid crystalline phases; nematics are complex liquids with a certain degree of long-range orientational order. That is, the constituent molecules are typically rod-like or elongated, and these elongated molecules flow about freely as in a conventional liquid but, whilst flowing, they tend to align along certain distinguished directions (*cf.* [10,50]).

There are several competing mathematical theories for nematic liquid crystals in the literature, such as the Doi–Onsager theory proposed by Doi [11] in 1986 and Onsager [38] in 1949, the Oseen–Frank theory proposed by Oseen [39] in 1933 and Frank [16] in 1958, the Ericksen–Leslie theory proposed by Ericksen [13] in 1961 and Leslie [27] in 1968, and the Landau–de Gennes theory proposed by Gennes [17] in 1995. The first one is a molecular kinetic theory, and the remaining three are continuum macroscopic theories. These theories can be derived or related to each other, under some assumptions. For instance, Kuzzu–Doi [26] and E–Zhang [12] formally derived the Ericksen–Leslie equation from the Doi–Onsager equations by taking small Deborah number limit. Wang–Zhang–Zhang [53] justified this formal derivation before the first singular time of the Ericksen–Leslie equations. Wang–Zhang–Zhang [54] presented a rigorous derivation of the Ericksen–Leslie equations from the Beris–Edwards model in the Landau–de Gennes framework. Ball–Majumdar [3] and Ball–Zarnescu [4] studied the differences and the overlap between the Oseen–Frank theory and the Landau–de Gennes theory. See [28–31] for further discussions.

Active hydrodynamics describe fluids with active constituent particles that have collective motion and are constantly maintained out of equilibrium by internal energy sources, rather than by the external forces applied to the system. In particular, when the particles have elongated shapes, usually the collective motion induces the particles to demonstrate orientational ordering at high concentration. Thus, there are natural analogies with nematic liquid crystals. Active hydrodynamics have wide applications and have attracted much attention in recent decades. For example, many biophysical systems are classified as active nematics, including microtubule bundles [47], cytoskeletal filaments [25], actin filaments [6], dense suspensions of microswimmers [55], bacteria [9], catalytic motors [42], and even nonliving analogues such as monolayers of vibrated granular rods [32]. For more information and discussions, see [5,11,20,23,24,43,45] and the references therein. Active nematic systems are distinguished from their well-studied passive counterparts since the constituent particles are active; that is, it is the energy consumed and dissipated by the active particles that drives the system out of equilibrium, rather than the external force applied at the boundary of the system, like a shear flow. Consequently, active dynamics are truly striking, and many novel effects have been observed in active systems, like the occurrence of giant density fluctuations [34,36,44], the spontaneous laminar flow [21,33,51], unconventional rheological properties [15,22,48], low Reynolds number turbulence [24,55], and very different spatial and temporal patterns compared to passive systems [8,18,34,35,46] arising from the interaction of the orientational order and the flow.

In this paper, we use the Landau–de Gennes Q-tensor description that is one of the most comprehensive descriptions, which describes the nematic state by a symmetric traceless 3×3 matrix, the Q-tensor order parameter with five independent degrees of freedom if the spatial dimension is three. A nematic phase is said to be (i) isotropic if $Q = 0$, (ii) uniaxial if Q has a pair of degenerate non-zero eigenvalues, and (iii) biaxial if Q has three distinct eigenvalues. In particular, a uniaxial phase has a single distinguished direction of nematic alignment, and a biaxial phase has a primary and secondary direction of preferred alignment. We remark that two-dimensional Q-tensors have been used to successfully model severely confined three-dimensional nematic systems that are effectively invariant in the third dimension.