Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations ••• (••••) •••–•••
Journal of Differential Equations
www.elsevier.com/locate/jde

Notes on the space–time decay rate of the Stokes flows in the half space ☆

Tongkeun Chang [a], Bum Ja Jin [b,*]

[a] CMAC Research Center, Yonsei University, Seoul 03722, South Korea
[b] Department of Mathematics Education, Mokpo National University, Muan-gun 58554, South Korea

Received 5 September 2016

Abstract

In this paper, a Stokes equations in the half space $\mathbb{R}^n_+$, $n \geq 2$ has been considered. We derive a rapid decay rate of the Stokes flow in space and time when the initial data decreases fast enough and satisfies some additional condition. Initial data decreasing too slowly to be $|x|h \in L^1(\mathbb{R}^n_+)$ are also considered.

MSC: primary 35K61; secondary 76D07

1. Introduction

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$, $n \geq 2$. In this paper, we consider the Stokes problem in the half space

$$\begin{aligned} &v_t - \Delta v + \nabla \pi = 0, div\, v = 0, \text{ in } \mathbb{R}^n_+ \times (0, \infty), \\ &v|_{t=0} = h, \quad v|_{x_n=0} = 0, \end{aligned} \tag{1.1}$$

☆ Tongkeun Chang was supported by NRF-2015R1A5A1009350 and Bum Ja Jin was supported by NRF-2016R1D1A1B03934133.
* Corresponding author.
E-mail addresses: chang7357@yonsei.ac.kr (T. Chang), bumjajin@mokpo.ac.kr (B.J. Jin).

http://dx.doi.org/10.1016/j.jde.2017.02.034

where $v = (v_1, \cdots, v_n)$ and π are unknown velocity and the pressure, respectively, $h = (h_1, \cdots, h_n)$ is the given initial value of the velocity. The initial data h satisfies the natural consistency condition that

$$\mathrm{div} h = 0 \text{ in } \mathbb{R}^n_+, \quad h_n|_{x_n=0} = 0. \tag{1.2}$$

Let $A = -\mathbb{P}\Delta$ denote the Stokes operator in $\mathbb{R}^n_+$, where $\mathbb{P} : L^r(\mathbb{R}^n_+) \to L^r_\sigma(\mathbb{R}^n_+)$, $1 < r < \infty$ is the projection operator onto the solenoidal vector fields. Define e^{-tA} be the semi-group operator generated by the Stokes operator A in the half space, then the solution of (1.1) can be expressed as $v = e^{-tA}h$.

The solution representation of the Stokes flow in the whole space is the same as the one of the heat flow with the same initial data. Hence, we have

$$\|\nabla^k e^{-tA}h\|_{L^q(\mathbb{R}^n)} \le ct^{-\frac{k}{2}-\frac{n}{2}+\frac{n}{2q}} \|h\|_{L^1(\mathbb{R}^n)} \tag{1.3}$$

with $k = 0, 1, \cdots$, and $1 \le q \le \infty$. For the initial data h decreasing fast enough at infinity, $\mathrm{div} h = 0$ in $\mathbb{R}^n$ implies $\int_{\mathbb{R}^n} h(y)dy = 0$. This leads to rapid decay rate of the Stokes flow as follows:

$$\|\nabla^k e^{-tA}h\|_{L^q(\mathbb{R}^n)} \le ct^{-\frac{k+1}{2}-\frac{n}{2}+\frac{n}{2q}} \||x|h\|_{L^1(\mathbb{R}^n)} \tag{1.4}$$

with $k = 0, 1, \cdots$, and $1 \le q \le \infty$. From (1.4) we observe that in L^∞ framework, the Stokes flow corresponding to initial data h with $|x|h \in L^1(\mathbb{R}^n)$ is dominated by $t^{-\frac{n+1}{2}}$ (from (1.3) we have similar observation). To be $|x|h \in L^1(\mathbb{R}^n)$, h should decrease sufficiently fast at infinity. For example, if h decrease as slowly as $|x|^{-n-1}$ at infinity, then $|x|h \notin L^1(\mathbb{R}^n)$. In [37], examples of h decreasing as slowly as $|x|^{-n-1}$ at infinity, but admitting Stokes flow dominated by $(1 + |x| + \sqrt{t})^{-n-1}$ are given. In [10], Brandolese derived a more rapid decay of Stokes flow and Navier–Stokes flow in space and time for the initial data h satisfying the following special conditions:

(a) h_j is odd in x_j and is even in each of the other variables,
(b) h is cyclically symmetric in the sense that

$$h_1(x_1, \cdots, x_n) = h_2(x_n, x_1, \cdots, x_{n-1}) = \cdots = h_n(x_2, \cdots, x_n, x_1).$$

See also [1,11,12,14,26,38,40] and references therein.

Concerning the Stokes flow and Navier–Stokes flow in the half space, temporal decay properties have been well studied. In [9], Borchers and Miyakawa showed that

$$\|\nabla^k e^{-tA}h\|_{L^q(\mathbb{R}^n_+)} \le ct^{-\frac{k}{2}-\frac{n}{2r}+\frac{n}{2q}} \|h\|_{L^r(\mathbb{R}^n_+)}, \ k = 0, 1, \cdots,$$

with $1 \le r < q \le \infty$ or $1 < r \le q < \infty$. In [20], Giga, Matsui and Shimizu proved that $\|\nabla e^{-tA}h\|_{L^1(\mathbb{R}^n_+)} \le ct^{-\frac{1}{2}}\|h\|_{L^1(\mathbb{R}^n_+)}$, and in [41], Shimizu proved that $\|\nabla e^{-tA}h\|_{L^\infty(\mathbb{R}^n_+)} \le ct^{-\frac{1}{2}}\|h\|_{L^\infty(\mathbb{R}^n_+)}$. In [18], Fujigaki and Miyakawa derived a rapid decay estimate in time such that