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Traveling wave to a reaction-hyperbolic system for axonal transport [☆]

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Abstract

In this paper, we study a class of nonlinear reaction-hyperbolic systems modeling the neuronal signal transfer in neuroscience. This reaction-hyperbolic system can be regarded as $n \times n$ $(n \geq 2)$ hyperbolic system with relaxation. We first prove the existence of traveling wave by Gershgorin circle theorem and mathematically describe the neuronal signal transport. Then for a special case $n = 2$, we show the traveling wave is nonlinearly stable, and obtain the convergence rate simultaneously by a weighted estimate.

Keywords: Reaction-hyperbolic system; Axonal transport; Traveling wave; Convergence rate

1. Introduction

In this paper, we consider a class of reaction-hyperbolic systems in one space dimension as follows

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$$\begin{cases} u_t^1 + \lambda_1 u_x^1 = \frac{1}{\epsilon} f_1(u^1, u^2), \\ u_t^2 + \lambda_2 u_x^2 = -\frac{1}{\epsilon} f_1(u^1, u^2) + \frac{1}{\epsilon} f_2(u^2, u^3), \\ \ldots\ldots \\ u_t^{n-1} + \lambda_{n-1} u_x^{n-1} = -\frac{1}{\epsilon} f_{n-2}(u^{n-2}, u^{n-1}) + \frac{1}{\epsilon} f_{n-1}(u^{n-1}, u^n), \\ u_t^n + \lambda_n u_x^n = -\frac{1}{\epsilon} f_{n-1}(u^{n-1}, u^n), \quad n \geq 2, \end{cases} \tag{1.1}$$

which model the axonal transport in neuroscience, see Reed and Blum [16]. Here u^i represents the i-th subpopulations, the term $\lambda_i u_x^i$ accounts for the transport of the i-th subpopulation with constant velocities λ_i satisfying $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Each f_i is a smooth function describing the biochemical processes of the constituents. The small parameter ϵ characterizes the fact that the biochemical process is much faster than the transport one.

The axonal transport is important for the maintenance and functions of nerve cells. The system (1.1) was introduced by Reed and Blum to model the axonal transport mathematically, see [2,3,14–16]. It was observed in experiments that the neuronal signal can be transferred through a jump of concentration of subpopulations, see [1,9]. This phenomenon can be explained in mathematics by the wave-like solution of the system (1.1). It was also observed by numerical simulations in [2,3,14–16] that these models exist wave-like solutions moving at a more or less constant velocity both in linear and nonlinear case, which is consistent with the experimental observation in neuroscience. However theoretically there is no traveling wave for the linear case, see [17]. Instead an approximate traveling wave was derived in [17], see also [3,7,8]. To our best knowledge, except the special case $n = 2$, there is no theoretical result on the traveling wave for the nonlinear cases which are more important in neuroscience, that is, in the system (1.1) f_i is nonlinear. For other interesting works, see [5,11] and the references therein.

In this paper, we focus on the traveling wave to the system (1.1). We first show the existence of traveling wave motivated by the kinetic theory for the Boltzmann equation. Precisely speaking, we observe that the equilibrium equation of the system (1.1) as $\epsilon \to 0$ is scalar conservation law in which there naturally has a shock wave. Based on this, we can expect a traveling wave exists for the system (1.1) with the same speed of the shock wave to the scalar conservation law, determined by the Rankine–Hugoniot condition. Before formulating our main results, we need the following assumption:

(A) The function $f_i(u^i, u^{i+1})$ is strictly decreasing with respect to u^i and strictly increasing with respect to u^{i+1}, i.e. $\frac{\partial f_i}{\partial u^i} < 0$ and $\frac{\partial f_i}{\partial u^{i+1}} > 0$. In addition $f_i(0, 0) = 0$.

The assumption **(A)** means that the chemical reactions form a chain and u^n is made from u^{n-1}, which is made from u^{n-2}. Note that the system (1.1) can be rewritten as

$$\begin{cases} \epsilon(u_t^1 + \lambda_1 u_x^1) = f_1(u^1, u^2), \\ \epsilon(u_t^2 + \lambda_2 u_x^2) = -f_1(u^1, u^2) + f_2(u^2, u^3), \\ \ldots\ldots \\ \epsilon(u_t^n + \lambda_n u_x^n) = -f_{n-1}(u^{n-1}, u^n), \quad n \geq 2, \end{cases} \tag{1.2}$$