ARTICLE IN PRESS
YJDEQ:8720

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics

Bernard Chazelle [a], Quansen Jiu [c], Qianxiao Li [b,*], Chu Wang [b,d]

[a] *Department of Computer Science, Princeton University, Princeton, NJ 08540, USA*
[b] *The Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08540, USA*
[c] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, PR China*
[d] *Nokia Bell Labs, 600-700 Mountain Avenue, Murray Hill, NJ 07974, USA*

Received 12 September 2016

Abstract

This paper establishes the global well-posedness of the nonlinear Fokker–Planck equation for a noisy version of the Hegselmann–Krause model. The equation captures the mean-field behavior of a classic multi-agent system for opinion dynamics. We prove the global existence, uniqueness, nonnegativity and regularity of the weak solution. We also exhibit a global stability condition, which delineates a forbidden region for consensus formation. This is the first nonlinear stability result derived for the Hegselmann–Krause model.

Keywords: Hegselmann–Krause model; Nonlinear Fokker–Planck equation; Well-posedness; Global stability

1. Introduction

Network-based dynamical systems feature agents that communicate via a dynamic graph while acting on the information they receive. These systems have received increasing attention lately because of their versatile use in modeling social and biological systems [1–7]. Typically, they consist of a fixed number N of agents, each one located at $x_k(t)$ on the real line. The agents'

* Corresponding author.
E-mail addresses: chazelle@cs.princeton.edu (B. Chazelle), jiuqs@cnu.edu.cn (Q. Jiu), qianxiao@math.princeton.edu (Q. Li), chu.wang@nokia.com (C. Wang).

http://dx.doi.org/10.1016/j.jde.2017.02.036
0022-0396/

Please cite this article in press as: B. Chazelle et al., Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.02.036

positions evolve as interactions take place along the edges of a dynamical graph that evolves endogenously. The motivation behind the model is to get a better understanding of the dynamics of collective behavior. Following [8,9], we express the system as a set of N coupled stochastic differential equations:

$$\mathrm{d}x_i = -\frac{1}{N}\sum_{j=1}^{N} a_{ij}(x_i - x_j)\mathrm{d}t + \sigma \mathrm{d}W_t^{(i)}, \tag{1}$$

where σ is the magnitude of the noise, $W_t^{(i)}$ are independent Wiener processes, and the "influence" parameter a_{ij} is a function of the distance between agents i and j; in other words, $a_{ij} = a(|x_i - x_j|)$, where a is nonnegative (to create attractive forces) and compactly supported over a fixed interval (to keep the range of the forces finite). Intuitively, the model mediates the competing tension between two opposing forces: the sum in (1) pulls the agents toward one another while the diffusion term keeps them jiggling in a Brownian motion; the two terms push the system into ordered and disordered states respectively. In the mean field limit, $N \to +\infty$, Equation (1) induces a nonlinear Fokker–Planck equation for the agent density profile $\rho(x,t)$ [8]:

$$\rho_t(x,t) = \left(\rho(x,t)\int \rho(x-y,t)ya(|y|)\mathrm{d}y\right)_x + \frac{\sigma^2}{2}\rho_{xx}(x,t). \tag{2}$$

The function $\rho(x,t)$ is the limit density of $\rho^N(x,t) := \frac{1}{N}\sum \delta_{x_j(t)}(dx)$, as N goes to infinity, where $\delta_x(dx)$ denotes the Dirac measure with point mass at x. The Fokker–Planck equation is a basic model in many areas of physics and it is a deterministic one describing how probability density functions evolve in time. A number of mathematical results, including well-posedness theory and the convergence to steady state as time $t \to \infty$, has been obtained for linear and nonlinear variants of the Fokker–Planck equation (see [10–17] and references therein).

This paper is concerned with the nonlinear, nonlocal Fokker–Planck equation (2) governing the density evolution of the noisy Hegselmann–Krause (*HK*) model. In the classic Hegselmann–Krause model, one of the most popular systems in consensus dynamics [18–20], each one of the N agents moves, at each time step, to the mass center of all the others within a fixed distance. The position of an agent represents its "opinion". If we add noise to this process, we obtain the discrete-time version of (1) for $a(y) = \mathbf{1}_{[0,R]}(y)$, where $\mathbf{1}_A$ is the usual set indicator function, equal to 1 if $y \in A$ and 0 otherwise. To be exact, the original *HK* model does not scale a_{ij} by $1/N$ but by the reciprocal of the number agents within distance R of agent i. Canuto et al. [21] have argued that this difference has a minor impact on the dynamics. By preserving the pairwise symmetry among the agents, however, the formulation (1) simplifies the analysis.

The *HK* model has been the subject of extensive investigation. A sample of the literature includes work on convergence and consensus properties [1,22,18,23–27], conjectures about the spatial features of the attractor set [28], and various extensions such as *HK* systems with inertia [29], leaders [30–32], or random jumps [33].

Some mathematical models for opinion formation have been proposed under different assumptions. A class of kinetic models of opinion formation and its asymptotic limit Fokker–Planck type equations are derived in [34], based on two-body interactions involving compromise and diffusion properties in exchanges between individuals. Later, based on [34], a mathematical model for opinion formation in a society that is built of two groups, one groups of "ordinary" people and one group of "strong opinion leaders", and its corresponding Fokker–Planck type

Please cite this article in press as: B. Chazelle et al., Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.02.036