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Relative equilibria with holes for the surface quasi-geostrophic equations

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Abstract

We study the existence of doubly connected rotating patches for the inviscid surface quasi-geostrophic equation left open by de la Hoz, Hassainia and Hmidi in 2016 in [10]. By using the approach proposed in 2016 by Castro, Córdoba and Gomez-Serrano in [4] we also prove that close to the annulus the boundaries are actually analytic curves.

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1. Introduction

In this paper we investigate the surface quasi-geostrophic (SQG) model which describes the evolution of the potential temperature θ according to the transport equation,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u = -\nabla^\perp (-\Delta)^{-\frac{1}{2}} \theta, \\ \theta_{|t=0} = \theta_0 \end{cases} \tag{1.1}$$

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where u refers to the velocity field and $\nabla^{\perp} = (-\partial_2, \partial_1)$. The operator $(-\Delta)^{-\frac{1}{2}}$ is defined as follows

$$(-\Delta)^{-\frac{1}{2}}\theta(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\theta(y)}{|x-y|}dy.$$

This model is used to study the atmospheric circulations near the tropopause and the ocean dynamics in the upper layers, see for instance [12,17,21]. This nonlinear transport equation is more singular than the vorticity equation for the 2D Euler equations where the connection between the velocity and the vorticity is given by the Biot–Savart law

$$u = -\nabla^{\perp}(-\Delta)^{-1}\theta.$$

Another model appearing in the literature which interpolates between the (SQG) and Euler equations is the $(\mathrm{SQG})_\alpha$ model, see [7], where the velocity is given by

$$u = -\nabla^{\perp}(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \quad \alpha \in (0, 2).$$

These equations have been intensively studied during the past few decades and abundant results have been established in different topics such as the well-posedness problem or the vorticity dynamics. For instance, it is well-known that for Euler equations when the initial data θ_0 belongs to $L^\infty \cap L^1$ then there is a unique global weak solution $\theta \in L^\infty(\mathbb{R}^+; L^\infty \cap L^1)$. This result is due to Yudovich, see for instance [27]. This theory fails for $\alpha > 0$ due to the singularity of the kernel. However, the local well-posedness can be elaborated in the sub-class of the vortex patches as it was shown in [5] and [11]. Recall that an initial datum is a vortex patch when it takes the form χ_D, which is the characteristic function of a smooth bounded domain D. The solutions keep this structure for a short time, that is, $\theta(t) = \chi_{D_t}$ where D_t is another domain describing the deformation of the initial one in the complex plane. The global existence of these solutions is an outstanding open problem except for Euler equations in which case Chemin proved in [6] the persistence of smooth regularity globally in time. Note that a significant progress towards settling this problem, for α enough close to zero, has been done recently in [20]. Another direction related to the construction of periodic global solutions through the bifurcation theory has been recently investigated. They correspond to rotating patches also called V-states or relative equilibria. In this setting the domain of the patch is explicitly given by a pure rotation with uniform angular velocity, that is, $D_t = R_{x_0,\Omega t}D$ where $R_{x_0,\Omega t}$ is the planar rotation with the center x_0 and the angle Ωt; the parameter Ω is the angular velocity. The first example of rotating patches goes back for Euler equation to Kirchhoff who discovered that an ellipse of semi-axes a and b rotates uniformly with the angular velocity $\Omega = \frac{ab}{(a^2+b^2)}$; see for instance [1, p. 304] and [22, p. 232]. One century later, Deem and Zabusky gave in [9] numerical evidence of the existence of the V-states with m-fold symmetry for each integer $m \in \{2, 3, 4, 5\}$ and afterwards Burbea gave an analytically proof in [2]. The main idea of the demonstration is to reformulate the V-states equations with the contour dynamics equations, using the conformal parametrization Φ, and to implement some bifurcation arguments. The bifurcation from the ellipses to countable curves of nonsymmetric rotating patches was discussed numerically and analytically in [4,15,18]. On the other hand we point out that the extension of this study to the $(\mathrm{SQG})_\alpha$ was successfully carried out in [3,13]. Moreover the boundary regularity was achieved in [3,4,16].