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Remarks on endpoint Strichartz estimates for Schrödinger equations with the critical inverse-square potential

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Abstract

The purpose of this paper is to study the validity of global-in-time Strichartz estimates for the Schrödinger equation on $\mathbb{R}^n$, $n \geq 3$, with the negative inverse-square potential $-\sigma|x|^{-2}$ in the critical case $\sigma = (n-2)^2/4$. It turns out that the situation is different from the subcritical case $\sigma < (n-2)^2/4$ in which the full range of Strichartz estimates is known to be hold. More precisely, splitting the solution into the radial and non-radial parts, we show that (i) the radial part satisfies a weak-type endpoint estimate, which can be regarded as an extension to higher dimensions of the endpoint Strichartz estimate with radial data for the two-dimensional free Schrödinger equation; (ii) other endpoint estimates in Lorentz spaces for the radial part fail in general; (iii) the non-radial part satisfies the full range of Strichartz estimates.

MSC: primary 35Q41; secondary 35B45

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1. Introduction

This paper is concerned with global-in-time dispersive properties of the unitary group e^{-itH} for the Schrödinger operator with the inverse-square potential of the form

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$$H := -\Delta - C_{\mathrm{H}}|x|^{-2}, \quad x \in \mathbb{R}^n,$$

where $n \geq 3$, $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ is the Laplacian and

$$C_{\mathrm{H}} := \frac{(n-2)^2}{4}$$

is the best constant in Hardy's inequality:

$$C_{\mathrm{H}} \int |x|^{-2}|u|^2 dx \leq \int |\nabla u|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^n). \tag{1.1}$$

In particular, we are interested in the validity of Strichartz estimates for e^{-itH}.

Let us first recall the free case. It is well known (see [45,23,49,28]) that the free Schrödinger evolution group $e^{it\Delta}$ satisfies the following family of space–time inequalities, known as *homogeneous and inhomogeneous Strichartz estimates*, respectively:

$$||e^{it\Delta}\psi||_{L^p(\mathbb{R};L^q(\mathbb{R}^n))} \leq C||\psi||_{L^2(\mathbb{R}^n)}, \tag{1.2}$$

$$\Big|\Big| \int_0^t e^{i(t-s)\Delta} F(s)ds \Big|\Big|_{L^p(\mathbb{R};L^q(\mathbb{R}^n))} \leq C||F||_{L^{\tilde{p}'}(\mathbb{R};L^{\tilde{q}'}\mathbb{R}^n))} \tag{1.3}$$

for admissible pairs (p,q) and $(\tilde{p},\tilde{q})$, where $p' = p/(p-1)$ denotes the Hölder conjugate of p, and (p,q) is said to be an (n-dimensional) admissible pair if

$$p,q \geq 2, \quad 2/p + n/q = n/2, \quad (p,q,n) \neq (2,\infty,2). \tag{1.4}$$

The condition (1.4) is necessary and sufficient for the validity of (1.2). In particular, as shown by [37,47], the two-dimensional endpoint estimates do not hold in general. The proof of (1.2) and (1.3) is based on the dispersive estimate of the form

$$||e^{it\Delta}\psi||_{L^\infty(\mathbb{R}^n)} \leq C|t|^{-n/2}||\psi||_{L^1(\mathbb{R}^n)}, \quad t \neq 0. \tag{1.5}$$

In fact, (1.5) together with the mass conservation $||e^{it\Delta}\psi||_{L^2} = ||\psi||_{L^2}$ implies both (1.2) and (1.3) for all admissible pairs (see [28]). It is worth noting that the Strichartz estimates play an important role in the study of the well-posedness and scattering theory for nonlinear Schrödinger equations (see, *e.g.*, monographs [9,48]).

There is also a vast literature on Strichartz estimates for Schrödinger equations with potentials (see [42,5,6,16,17,24,12,34,13,2,31,4] and references therein). In particular, the Schrödinger operator with inverse-square potentials of the form $H_\sigma = -\Delta - \sigma|x|^{-2}$ with $\sigma \in \mathbb{R}$ has recently been extensively studied since it represents a borderline case as follows. Both the decay rate $|x|^{-2}$ as $|x| \to +\infty$ and the local singularity $|x|^{-2}$ as $|x| \to 0$ are critical for the validity of Strichartz and dispersive estimates (see [25,14]). Moreover, the case with $\sigma = C_{\mathrm{H}}$, *i.e.*, $H_\sigma = H$, is also critical in the following sense. On one hand, if $\sigma > C_{\mathrm{H}}$, H_σ is not lower semi-bounded (due to the fact that C_{H} is the best constant in (1.1)) and has infinitely many negative eigenvalues diverging to $-\infty$ (see [39]). In particular, there is no hope to obtain any kind of global-in-time dispersive estimates. On the other hand, if $\sigma < C_{\mathrm{H}}$, the full range of Strichartz estimates is