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Quasi-periodic response solutions in forced reversible systems with Liouvillean frequencies ☆

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Abstract

This paper is concerned with the existence of quasi-periodic response solutions for a class of reversible forced harmonic oscillators with two basic frequencies $\omega = (1, \alpha)$, α is an irrational number. Since we do not impose usual Diophantine or Brjuno conditions on ω, it can also be Liouvillean. The proof is based on a modified KAM (Kolmogorov–Arnold–Moser) theorem for reversible systems.

MSC: 37J40; 34C27; 34C15

Keywords: KAM theory; Reversible vector field; Quasi-periodic solutions

1. Introduction and main result

1.1. Introduction

A variety of nonlinear vibration problems arising from the physical real word can be characterized by the forced oscillators

$$\ddot{x} + c\dot{x} + ax = f(t, x, \dot{x}), \ x \in \mathbb{R}, \tag{1.1}$$

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where c and $a > 0$ are parameters. The forcing term f is quasi-periodic in t with n real, rationally independent frequencies $\omega_1, \cdots, \omega_n$. That is $f(t, x, \dot{x})$ can be rewritten as $f(t, x, \dot{x}) = F(\omega_1 t, \cdots, \omega_n t, x, \dot{x})$, where the function $F(\theta_1, \cdots, \theta_n, x, \dot{x})$ is continuous and 2π-periodic in all $\theta_1, \cdots, \theta_n$. The Duffing and the van der Pol equations with quasi-periodic forcing are special cases of equation (1.1). From both the mathematical and the physical points of view, an important and interesting question is whether equation (1.1) possess response solutions, i.e., solutions that are quasi-periodic with the same frequencies as forcing term f.

If $|c|$ is large compared to the size of f, J. Stoker [9] gave an existence proof for the equation above by classical contraction principle. If $|c|$ is small compared to the size of f, however, this method fails due to the difficulty of small divisors. And Stoker further proposed whether such solutions still exist in this case. The first breakthrough in this problem was made by J. Moser [7]. He considered a modification of equation (1.1):

$$\ddot{x} + ax = \varepsilon f(\omega t, x, \dot{x}), (0 < \varepsilon < 1), \tag{1.2}$$

which is reversible with respect to the involution $S(x, y) = (x, -y)$, i.e.,

$$f(\theta, x, y) = f(-\theta, x, -y). \tag{1.3}$$

By KAM theory, Moser obtained the existence of response solutions, thus gave a positive answer to Stoker's problem for $c = 0$. Here the reversible condition (1.3) is necessary to keep $a = a(\varepsilon)$ real in the proof. We mentioned that this is the first KAM result for (non-Hamiltonian) reversible systems. Later, M. Friedman [3] and B. Braaksma and H. Broer [2] also solved the problem for small non-zero $|c|$ by KAM method and by normal form method. In their proofs, parameter c and a are needed to guarantee the existence of solutions, where parameter c is used to control the normal hyperbolicity of torus. In [2,3,7], to handle small divisors problem, frequency vector $\omega := (\omega_1, \cdots, \omega_n)$ is required to be Diophantine:

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{|k|^{\tau}}, \text{ for all } k \in \mathbb{Z}^n \setminus \{0\}, \tag{1.4}$$

for some $\tau > n - 1$ and $\gamma > 0$, where $\langle k, \omega\rangle := \sum_{i=1}^{n} k_i \omega_i$, $|k| := \sum_{i=1}^{n} |k_i|$. G. Gentile [4,5] studied, by Lyapunov–Schmidt reduction, the existence of response solutions for forced strongly dissipative systems under Brjuno condition, where the nonlinearity has the form of $f(\omega t, x, \dot{x}) = h(\omega t) - g(x) = \sum_{k\in\mathbb{Z}^d} h_k e^{i\langle k, \omega t\rangle} - g(x)$, and the equation $h_0 - g(x)$ has a real zero of odd order.

Some recent results in reducibility theory for linear quasi-periodic systems show that it is possible to develop Liouvillean (ω is called Liouvillean if it is not Diophantine but rationally independent) KAM theory for some special systems. In [1], using CD bridge technique, A. Avila, B. Fayad and R. Krikorian developed a new KAM scheme for $SL(2, \mathbb{R})$ cocycles with one Liouvillean frequency. X. Hou and J. You [6] further studied the following real analytic linear quasi-periodic systems with two frequencies in $sl(2, \mathbb{R})$:

$$\begin{cases} \dot{x} = A(\theta)x, \\ \dot{\theta} = \omega = (1, \alpha). \end{cases} \tag{1.5}$$

The most important results in [6] are the almost reducibility and the rotations reducibility of system (1.5) via two KAM schemes without any arithmetic restrictions on irrational frequency ω,