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Existence, uniqueness and regularity results on nonlocal balance laws

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Abstract

We consider a class of nonlocal balance laws as initial value problems on a finite time horizon and show existence and uniqueness of the corresponding weak solutions. The description "nonlocal" refers to the velocity of the balance law that depends on the weighted integral over an area in space at any given time. Existence of a weak solution for initial data and right hand side data in $L^1 \cap L^\infty$, in L^∞ and in special cases in L^1 is shown via the method of characteristics, resulting in a fixed-point problem in the nonlocal term. The uniqueness of a weak solution with relatively weak assumptions on the flux function and the nonlocal term is established, so that the uniqueness result does not require the well-known "Kružkov" entropy condition as it is typical for (local) balance laws and was up to now used in the available literature also for nonlocal balance laws.

MSC: 35L03; 35L65; 65M25; 35D30

Keywords: Nonlocal conservation law; Nonlocal balance law; Existence; Uniqueness; Weak solution; Regularity

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1. Introduction and problem formulation

In the following work, we consider the nonlocal balance law in conservative form in the time and space dependent variable $q : \Omega_T \to \mathbb{R}$ on $\Omega_T := [0, T] \times \mathbb{R}$ for $T \in \mathbb{R}_{>0}$ as given by

$$q_t(t,x) + \partial_x\Big(\lambda\Big[W\big[q,\gamma,a,b\big]\Big](t,x)q(t,x)\Big) = h(t,x) \qquad (t,x) \in \Omega_T \tag{1}$$

$$q(0,x) = q_0(x) \qquad x \in \mathbb{R} \tag{2}$$

supplemented by the nonlocal term W

$$W[q,\gamma,a,b]\big(t,x\big) := \int_{a(x)}^{b(x)} \gamma(t,x,y)q(t,y)\,\mathrm{d}y \qquad (t,x) \in \Omega_T. \tag{3}$$

Thereby, the flux function $\lambda[W]$ is made more explicit in Assumption 2.1 below and sometimes also referred to as "velocity function". q_0 is the initial data and h represents the source term, while W is the so called nonlocal impact (in supply chain modeling this is called the "**W**ork in progress"). γ represents a weight of this nonlocal impact W in space and time, and finally a and b represent the "boundaries" of that nonlocal term in space. The complicated notation for λ involving W enables us to keep track of the involved functions in order to denote the results of this paper directly with respect to them.

As for well-posedness of related problems semi-group approaches or Lax–Friedrichs-type schemes have been used in the literature, where also the Kružkov entropy condition plays a prominent role. Thereby, the entropy condition is required to guarantee uniqueness [5,10,4], while in this contribution the problem is – in a more general form – investigated via the method of characteristics. We will show that the assumed Kružkov entropy condition [22] is obsolete for uniqueness of the weak solution under relatively weak assumptions and that any solution satisfying the somewhat seminal framework, we have introduced in Equations (1)–(3), has to be unique as long as the solution does not blow up.

Thus, this contribution improves, simplifies and generalizes several of the already available results significantly.

In the proposed framework, no shocks can emerge – or more precisely – in case a shock emerges, the solution breaks down by necessity.

We will present an (explicit) formula for the solution, involving a fixed-point equation in the nonlocal term. This formula also enables us to study the regularity of solutions depending on the regularity of initial data q_0, velocity λ, weight γ and boundary terms a, b for the nonlocal term, although the explicit dependency of the nonlocal term with respect to the spatial variable makes the analysis much harder.

In [31], the authors study a nonlocal conservation law modeling particle suspension, in a different framework and for a specific nonlocal term ($\lambda[w] \equiv w$) involving a kernel. It is worth mentioning that they also perform a fixed-point argument but in the solution and not in the nonlocal term as performed here. They also use the method of characteristics and mention that "... no entropy condition is necessary for uniqueness, supporting the claim that ... has essentially linear regularity properties" [31, Remark 2.4.1].

In [1], the existence and uniqueness of a solution to a "nonlocal–local" (i.e. a local conservation law incorporating a nonlocal term) conservation law is shown. Due to the involvement of

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