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# On partial regularity problem for 3D Boussinesq equations

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**Abstract**

In this paper, we study the partial regularity of the solutions for the three-dimensional Boussinesq equations. We first prove a criterion of local Hölder continuous of the suitable weak solutions of the Boussinesq equations, and show that one-dimensional Hausdorff measure of the singular point set is zero. Secondly, we present a local uniform gradient estimate on the suitable weak solutions and assert that the local behavior of the solution can be dominated by some scaled quantities, such as the scaled local $L^3$-norm of the velocity. Besides, when the initial data $v_0$ and $\theta_0$ decay sufficiently rapidly at $\infty$, the distribution of the regular point set of the suitable weak solutions is also considered. Based on it, one can find that MHD equations are more similar to Navier–Stokes equations than Boussinesq equations. Finally, we give a local regularity criterion of the suitable weak solutions near the boundary.


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## 1. Introduction

In this paper, we study the local properties of weak solutions of the three-dimensional Boussinesq equations

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$$\begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p = \Delta v + \theta \mathrm{e}_3, \\ \frac{\partial \theta}{\partial t} + (v \cdot \nabla)\theta - \Delta\theta = 0, \\ \nabla \cdot v = 0, \\ v(x,0) = v^0, \theta(x,0) = \theta^0, \end{cases} \tag{1.1}$$

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x,t)$, $j = 1, 2, 3$, is the velocity vector field with $(x,t) \in D := \Omega \times ]0, T[$, where $\Omega$ is a open domain of $\mathbb{R}^3$, and $T$ is any fixed positive constant. The notation $p = p(x,t)$ is the scalar pressure, $\theta(x,t)$ is the scalar temperature and $\mathrm{e}_3 = (0, 0, 1)$. Our interest is the initial boundary value problem on $\Omega$, in addition to (1.1) on $\Omega \times ]0, T[$, one requires

$$\begin{cases} (v,\theta)(x,0) = (v_0, \theta_0)(x), \quad x \in \Omega, \\ v(x,t) = 0, \ \theta(x,t) = 0 \quad x \in \partial\Omega, \quad 0 < t < T. \end{cases} \tag{1.2}$$

The Boussinesq system (1.1) is a simple model widely used in the modeling of oceanic and atmospheric motions and play an important role in the atmospheric sciences (see e.g. [16]). These models also appear in many other physical problems. We refer for instance to [1,2] for more details.

The problem (1.1)–(1.2) with $\theta = 0$ is the initial value problem to the Navier–Stokes equations, the existence of global weak solutions of the Navier–Stokes equations was shown by Leray and Hopf in their pioneering works [12,14] long ago. When the spatial dimension is three, a large gap remains between the regularity available in the existence results and additional regularity required in the sufficient conditions to guarantee the smoothness of weak solutions. In order to narrow this gap, great efforts has been made by many authors, such as Scheffer in [19], Caffarelli, Kohn and Nirenberg in [3], Lin in [15], Tian and Xin in [23], see also the reference [13,22] therein. In particular, Caffarelli et al. in [3] proved that there exists a suitable weak solution (the concept of the suitable weak solutions was first introduced in [18]) to the Navier–Stokes equations $v$ and $p$, and one-dimensional Hausdorff measure of the possible space–time singular points set (denoted by $S$) is zero, where $S$ is defined by $S \equiv \{(x,t) \in D : v \text{ is not } L^\infty \text{ in any neighborhood of } (x,t)\}$. After that Lin give a new proof of the partial regularity of weak solution of the Navier–Stokes equations with the external force $f = 0$, and by proving the Hölder continuous of the weak solution in [15] to get the possible space–time singular points set is zero. Furthermore, Ladyzhenskaya and Seregin in [13] considered the more generalized case, namely, the external force $f \neq 0$, and get the same result as Lin. Through the gradient estimates of the suitable weak solution of the Navier–Stokes equations, Tian and Xin in [23] also got the possible space–time singular points set is zero. Does the Caffarelli–Kohn–Nirenberg condition imply Hölder continuity of $v$ in a neighborhood of $z_0$, if $x_0$ belongs to the boundary $\partial\Omega$? The author shown that the answer to above question is positive if there exists a neighborhood $\mathcal{O}$ of the point $x_0$ such that $\Gamma = \partial\Omega \cap \mathcal{O}$ lies in a hyperplane and $v = 0$ on $\Gamma \times ]0, T[$ in [20], in which the crucial estimates near the boundary to the solution of non-stationary linearized Navier–Stokes equations (see [21]) was applied by the author.

As to the Boussinesq equations, Guo and Yuan defined a kind of "suitable" weak solution to the Boussinesq equations in [7] and [8] respectively. By using some additional assumptions, such as $\theta_0 \in L^\infty$ (which ensures $\theta \in L^\infty$), and by proving the local boundness of the "suitable" weak solution, the authors got the one-dimensional Hausdorff measure for the set of the possible space–time singular points of is zero. Motivated by the work of Lin [15], Ladyzhenskaya and