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Lipschitz regularity results for nonlinear strictly elliptic equations and applications

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Abstract

Most of Lipschitz regularity results for nonlinear strictly elliptic equations are obtained for a suitable growth power of the nonlinearity with respect to the gradient variable (subquadratic for instance). For equations with superquadratic growth power in gradient, one usually uses weak Bernstein-type arguments which require regularity and/or convex-type assumptions on the gradient nonlinearity. In this article, we obtain new Lipschitz regularity results for a large class of nonlinear strictly elliptic equations with possibly arbitrary growth power of the Hamiltonian with respect to the gradient variable using some ideas coming from Ishii–Lions' method. We use these bounds to solve an ergodic problem and to study the regularity and the large time behavior of the solution of the evolution equation.

MSC: primary 35J60, 35K55; secondary 35D40, 35B65, 35B45, 35B51, 35B40, 35B05

Keywords: Nonlinear strictly elliptic equations; Nonlinear parabolic equations; Hamilton–Jacobi equations; Lipschitz estimates; Asymptotic behavior; Viscosity solutions

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1. Introduction

The main goal of this work is to obtain gradient bounds, which are uniform in $\epsilon > 0$ and t respectively, for the viscosity solutions of a large class of nonlinear strictly elliptic equations

$$\epsilon v^\epsilon - \mathrm{trace}(A(x)D^2 v^\epsilon) + H(x, Dv^\epsilon) = 0,\ x \in \mathbb{T}^N, \tag{1.1}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{trace}(A(x)D^2 u) + H(x, Du) = 0, & (x,t) \in \mathbb{T}^N \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{T}^N. \end{cases} \tag{1.2}$$

We work in the periodic setting ($\mathbb{T}^N$ denotes the flat torus $\mathbb{R}^N/\mathbb{Z}^N$) and assume for simplicity that $A(x) = \sigma(x)\sigma(x)^T$ with $\sigma \in W^{1,\infty}(\mathbb{T}^N; \mathcal{M}_N)$. Let us mention that all the results of this paper hold true if $\sigma \in C^{0,1/2}(\mathbb{T}^N; \mathcal{M}_N)$.

We recall that a diffusion matrix A is called strictly elliptic if

$$\text{there exists } \nu > 0 \text{ such that } A(x) \geq \nu I, \quad x \in \mathbb{T}^N. \tag{1.3}$$

Most of Lipschitz regularity results for elliptic equations are obtained for a suitable growth power with respect to the gradient variable (subquadratic for instance, see Frehse [14], Gilbarg–Trudinger [15]). In this article, we establish some gradient bounds

$$|Dv^\epsilon|_\infty \leq K, \ \text{ where } K \textit{ is independent of } \epsilon, \tag{1.4}$$

$$|Du(\cdot,t)|_\infty \leq K, \ \text{ where } K \textit{ is independent of } t, \tag{1.5}$$

for strictly elliptic equations whose Hamiltonians H have arbitrary growth power in the gradient variable, which is unusual.

An important feature of our work is that we look for uniform gradient bounds in ϵ or t. In many results, the bounds depend crucially on the L^∞ norm of the solution (which looks like $O(\epsilon^{-1})$ or $O(t)$), something we want to avoid in order to be able to solve some ergodic problems by sending $\epsilon \to 0$ or to study the large time behavior of $u(x,t)$ when $t \to +\infty$. These applications are discussed more in details below and are done in Section 4. We focus now on the more delicate part, i.e., the Lipschitz bounds for (1.1).

Let us start by recalling the existing results when H is superquadratic and coercive. Hölder regularity of the solution is proved under the very general assumption

$$H(x,p) \geq \frac{1}{C}|p|^k - C, \qquad \text{with } k > 2,$$

see Capuzzo Dolcetta et al. [10], Barles [7], Cardaliaguet–Silvestre [11], Armstrong–Tran [3]. But there are only few results as far as Lipschitz regularity is concerned. In general they are established using Bernstein method [15,19] or the adaptation of this method in the context of viscosity solutions, see Barles [5], Barles–Souganidis [8], Lions–Souganidis [21], Capuzzo Dolcetta et al. [10]. This approach requires some structural assumptions on H which are often close to

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