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Well-posedness for a class of doubly nonlinear stochastic PDEs of divergence type ☆

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Abstract

We prove well-posedness for doubly nonlinear parabolic stochastic partial differential equations of the form $dX_t - \operatorname{div}\gamma(\nabla X_t)\,dt + \beta(X_t)\,dt \ni B(t, X_t)\,dW_t$, where γ and β are the two nonlinearities, assumed to be multivalued maximal monotone operators everywhere defined on $\mathbb{R}^d$ and $\mathbb{R}$ respectively, and W is a cylindrical Wiener process. Using variational techniques, suitable uniform estimates (both pathwise and in expectation) and some compactness results, well-posedness is proved under the classical Leray–Lions conditions on γ and with no restrictive smoothness or growth assumptions on β. The operator B is assumed to be Hilbert–Schmidt and to satisfy some classical Lipschitz conditions in the second variable.

MSC: 60H15; 35R60; 35K55; 35D30; 47H05; 46N30

Keywords: Doubly nonlinear stochastic equation; Divergence; Variational approach; Existence of solutions; Continuous dependence; Multiplicative noise

1. Introduction

In this work, we consider the boundary value problem with homogeneous Dirichlet conditions associated to a doubly nonlinear parabolic stochastic partial differential equation on an smooth bounded domain $D \subseteq \mathbb{R}^d$ of the type

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$$dX_t - \operatorname{div}\gamma(\nabla X_t)\,dt + \beta(X_t)\,dt \ni B(t, X_t)\,dW_t \quad \text{in } D \times (0,T)\,, \tag{1.1}$$

$$X(0) = X_0 \quad \text{in } D\,, \tag{1.2}$$

$$X = 0 \quad \text{on } \partial D \times (0,T)\,, \tag{1.3}$$

where γ and β are two maximal monotone operators everywhere defined on $\mathbb{R}^d$ and $\mathbb{R}$, respectively, W is a cylindrical Wiener process, and B is a random time-dependent Hilbert–Schmidt operator (we will state the complete assumptions on the data in the next section). We prove existence of global solutions as well as a continuous dependence result using variational techniques (see e.g. the classical works [17,22,23] about the variational approach to SPDEs).

The problem (1.1)–(1.3) is very interesting from the mathematical point of view: as a matter of fact, the equation presents two strong nonlinearities. The first one is represented by γ within the divergence operator: in this case, we will need to assume some classical growth assumptions (the so-called Leray–Lions conditions) in order to recover a suitable coercivity on a natural Sobolev space. The other nonlinearity is represented by the operator β: this is treated as generally as possible, with no restriction on the growth and regularity. Because of this generality, the concept of solution and the appropriate estimates are more difficult to achieve, as we will see. We point also out that dealing with maximal monotone graphs makes our analysis absolutely exhaustive. As a matter of fact, in this way we include in our treatment any continuous increasing function β (with any order of growth), as well as every increasing function with a countable number of jumps: indeed, it is a standard matter to see that if β is an increasing function on $\mathbb{R}$ with jumps in $\{x_n\}_{n\in\mathbb{N}}$, one can obtain a maximal monotone graph by setting $\beta(x_n) = [\beta_-(x_n), \beta_+(x_n)]$. Finally, very mild assumptions on the noise are required, so that our results fit to any reasonable random time-dependent Hilbert–Schmidt operator B; in the case of multiplicative noise, only classical Lipschitz continuity hypotheses are in order.

The noteworthy feature of this paper is that problem (1.1)–(1.3) is very general and embraces a wide variety of specific sub-problems which are interesting on their own: consequently, we provide with our treatment a unifying analysis to several cases of parabolic SPDEs. Let us mention now about some of these and the main related literature.

If γ is the identity on $\mathbb{R}^d$, the resulting equation is the classical semilinear SPDE driven by the Laplace operator $dX - \Delta X\,dt + \beta(X)\,dt \ni B\,dW_t$, which has been widely studied. For example, in [21], global existence results of solutions are provided in the semilinear case, with the laplacian being generalized to any suitable linear operator: here, the idea is to doubly approximate the problem, in order to recover more regularity on β and B, to find then appropriate estimates on the approximated solutions and finally to pass to the limit in the equation. Moreover, in [13], mild solutions are obtained under the strong hypotheses that β is a polynomial of odd degree $m > 1$ and B can be written as $(-\Delta)^{-\frac{s}{2}}$ for a suitable s; in [3], existence of mild solutions is proved with no restrictive hypotheses on the growth of β, but imposing some strong continuity assumptions on the stochastic convolution. In [20], well-posedness is established for the semilinear problem in a L^q setting, with β having polynomial growth.

If γ is the monotone function on $\mathbb{R}^d$ given by $\gamma(x) = |x|^{p-2}x$, $x \in \mathbb{R}^d$, for a certain $p \geq 2$, then the term represented by the divergence in (1.1) is the usual p-laplacian: in this case, our equation becomes $dX - \Delta_p X\,dt + \beta(X)\,dt \ni B\,dW_t$, where $\Delta_p\cdot := \operatorname{div}(|\nabla\cdot|^{p-2}\nabla\cdot)$. This problem is far more interesting and complex than the semilinear case since $-\Delta_p$ is nonlinear for any $p > 2$ and consequently (1.1) becomes doubly nonlinear in turn. Among the great literature dealing with this problem, we can mention [18] for example, where the stochastic p-Laplace equation is studied in the singular case $p \in [1,2)$, and [19] as well.

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