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Growth models for tree stems and vines

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Abstract

The paper introduces a PDE model for the growth of a tree stem or a vine. The equations describe the elongation due to cell growth, and the response to gravity and to external obstacles. An additional term accounts for the tendency of a vine to curl around branches of other plants.

When obstacles are present, the model takes the form of a differential inclusion with state constraints. At each time t, a cone of admissible reactions is determined by the minimization of an elastic deformation energy. The main theorem shows that local solutions exist and can be prolonged globally in time, except when a specific "breakdown configuration" is reached. Approximate solutions are constructed by an operator-splitting technique. Some numerical simulations are provided at the end of the paper.

1. Introduction

We consider a simple mathematical model describing how the stem of a plant grows, and how it reacts to external constraints, such as branches of other plants. At each time t the stem is described by a curve $\gamma(t, \cdot)$ in 3-dimensional space. The model takes into account the linear elongation due to cell growth and the upward bending as a response to gravity. In the case of vines, an additional term accounts for the tendency to curl around branches of other plants.

From a theoretical perspective, the main challenge comes from the presence of external obstacles, resulting in a number of unilateral constraints. Ultimately, this yields a differential inclusion on a closed subset of $H^2([0, T]; \mathbb{R}^3)$. We remark that most of the literature on dif-

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ferential inclusions with constraints is concerned with the case where the cone of admissible reactions produced by the (possibly moving) obstacle is perpendicular to its boundary [3,4,6,7]. In Moreau's "sweeping process", this assumption plays an essential role in the proof of existence and continuous dependence of solutions. In our model, at a time when part of the stem touches the obstacle, the evolution is governed by the minimization of an instantaneous elastic deformation energy, subject to the external constraints. As a consequence, the cone of admissible velocities determined by the obstacle's reaction can be very different from the normal cone. In certain "breakdown configurations", as shown in Fig. 4, this cone of admissible reactions actually happens to be tangent.

Our main result, Theorem 1 in Section 3, establishes the local existence of solutions to the growth model with obstacles. These solutions can be extended globally in time, provided that a specific "breakdown configuration" is never reached. As already mentioned, since the cone of admissible reactions is not a normal cone, the uniqueness and continuous dependence of solutions is a difficult problem that requires a substantially different approach from [3,4,6,7]. A detailed analysis of this issue will appear in the forthcoming paper [2].

The remainder of this paper is organized as follows. Section 2 introduces the basic model and derives an evolution equation satisfied by the growing curve. If obstacles are present, this takes the form of a differential inclusion in the space $H^2([0,T];\ \mathbb{R}^3)$. This is supplemented by unilateral constraints, requiring that at all times the curve $\gamma(t,\cdot)$ remains outside a given set. In Section 3 we give a definition of solution and state the main existence theorem. Namely, solutions exist locally in time and can be prolonged up to the first time when a "breakdown configuration" is reached. A precise definition of these "bad" configurations is given at (3.9)–(3.10) and illustrated in Fig. 3. In essence, this happens when the tip of the stem touches the obstacle perpendicularly, and all the portions of the stem that do not touch the obstacle are straight segments.

The existence of solutions is proved in Sections 4 and 5, constructing a sequence of approximations by an operator-splitting technique. Each time step involves:

- a regular evolution operator, modeling the linear growth and the bending in response to gravity (possibly including also the curling of vines around branches of other plants),
- a singular operator, accounting for the obstacle reaction.

Much of the analytical work is carried out in Section 4, where we introduce a "push-out" operator and derive some key a priori estimates. Section 5 completes the proof of the main theorem. This is based on a compactness argument, which yields a convergent subsequence of approximate solutions.

In Section 6 we briefly describe how our results can be extended to more general models, including the case where the elastic energies associated with twisting and bending of the stem come with different coefficients. Finally, Section 7 presents some numerical simulations, in the case of one or two obstacles, in two space dimensions. The code used for these simulations can be downloaded at [8].

2. The basic model

We assume that new cells are generated at the tip of the stem, then they grow in size. At time $t \geq 0$, the length of the cells born during the time interval $[s,\ s+ds]$ is measured by

$$d\ell \;=\; (1-e^{-\alpha(t-s)})\,ds\,, \tag{2.1}$$