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Monotonicity formulas for extrinsic triharmonic maps and the triharmonic Lane–Emden equation

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Abstract

We derive monotonicity formulas and ε-regularity results for extrinsic triharmonic maps and triharmonic Lane–Emden equations. As an application, we prove partial regularity results for both types of equations in the supercritical regime.

MSC: 35B65; 35J48; 35J91

Keywords: Monotonicity formula; Triharmonic map; Triharmonic Lane–Emden equation; Partial regularity

1. Introduction

Ever since Peter Price found his monotonicity formula for Yang–Mills fields [20], monotonicity formulas are one of the most important tools for studying supercritical equations and systems. Since then, monotonicity formulas for stationary harmonic [23,9] and biharmonic maps [6], the Lane–Emden equation [10,19] and its biharmonic and fractional versions [8,7] were found. In [5], Colding discovered striking monotonicity formulas for the Green's function on non-parabolic manifolds with non-negative Ricci curvature which were further investigated in [4].

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One of the main motivations of this article is to extend these known monotonicity formulas to equations of order six. We will then use these new monotonicity formulas to derive partial regularity results. Surprisingly, we only succeed in deriving a useful monotonicity formula for stationary extrinsic triharmonic maps if the dimension is less than 20. Similarly, for the stationary solutions of the triharmonic Lane–Emden equation we end up with a restriction on the range of exponents depending on the dimension.

Let $N \subset \mathbb{R}^m$ be a compact embedded submanifold without boundary and let $\Omega \subset \mathbb{R}^n$ be open. We call a map $u : \Omega \to N$ weakly extrinsic triharmonic map if u is a critical point of the extrinsic triharmonic energy

$$E(u) := \int_\Omega |\nabla \Delta u|^3 dx \tag{1.1}$$

on the target manifold, i.e., if for all $\phi \in C_c^\infty(\Omega, \mathbb{R}^m)$ we have

$$\frac{d}{d\tau} E(\pi(u + \tau\phi))\Big|_{\tau=0} = 0. \tag{1.2}$$

Here, π denotes the nearest neighbor projection of an open neighborhood of N onto the manifold N and C_c^∞ denotes the space of smooth functions with compact support.

Riviere found examples of everywhere discontinuous weakly harmonic maps into $\mathbb{S}^3$ [21]. To get a satisfactory regularity results, one therefore has to restrict the class of solutions further. Usually one considers stationary harmonic maps, i.e. critical points of the Dirichlet energy

$$\int_\Omega |\nabla u|^2 dx,$$

for which also the inner variation of the energy vanishes. Analogously, we call a weakly extrinsic triharmonic map stationary if the inner variation of the triharmonic energy vanishes, i.e., if

$$\frac{d}{d\tau} E(u \circ (id_\Omega + \tau\phi))\Big|_{\tau=0} = 0 \tag{1.3}$$

for all $\phi \in C_c^\infty(\Omega, \mathbb{R}^n)$.

Following ideas from [6], we will show that for a stationary extrinsic triharmonic map the quantity

$$\frac{1}{2} r^{6-n} \int_{B_1(0)} |\nabla \Delta u|^2 dx + 3r^{5-n} \int_{\partial B_1(0)} |\Delta u|^2 dS + 4(n-4) r^{3-n} \int_{\partial B_1(0)} |\nabla u|^2 dS + DLOT$$

is monotonically decreasing if $4 \le n \le 20$. Here, the term $DLOT$ consists of special lower order terms and distributional derivatives thereof (cf. Corollary 3.8). This extends the formulas for harmonic maps [20] and biharmonic maps [6] to these equations of order six.

Combining this monotonicity formula with an ε-regularity result for weak extrinsic triharmonic maps into spheres (Proposition 4.3), we prove partial regularity for stationary extrinsic triharmonic maps into spheres.

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