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Multiple positive solutions to elliptic boundary blow-up problems

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Abstract

We prove the existence of multiple positive radial solutions to the sign-indefinite elliptic boundary blow-up problem

$$\begin{cases} \Delta u + \big(a^+(|x|) - \mu a^-(|x|)\big) g(u) = 0, & |x| < 1, \\ u(x) \to \infty, & |x| \to 1, \end{cases}$$

where g is a function superlinear at zero and at infinity, a^+ and a^- are the positive/negative part, respectively, of a sign-changing function a and $\mu > 0$ is a large parameter. In particular, we show how the number of solutions is affected by the nodal behavior of the weight function a. The proof is based on a careful shooting-type argument for the equivalent singular ODE problem. As a further application of this technique, the existence of multiple positive radial homoclinic solutions to

$$\Delta u + \big(a^+(|x|) - \mu a^-(|x|)\big) g(u) = 0, \qquad x \in \mathbb{R}^N,$$

is also considered.

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1. Introduction

In the qualitative theory of elliptic PDEs, problems of the type

$$\begin{cases} \Delta u + f(x,u) = 0, & x \in \Omega, \\ u(x) \to +\infty, & x \to \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, are usually referred to as *boundary-blow up* problems and they date back to Bieberback [9] and Rademacher [44], arising from questions in Geometry and Mathematical Physics. Further classical contributions were then given by Keller [30] and Osserman [40]; as for more recent works, we just quote [1,4,5,35], referring to the introductions of [25,36] for more information and references on the subject.

Our investigation is motivated by a recent paper by García-Melián [24], dealing with the existence of *positive* solutions to the boundary blow-up problem

$$\begin{cases} \Delta u + \big(\epsilon a^+(x) - a^-(x)\big)u^p = 0, & x \in \Omega, \\ u(x) \to +\infty, & x \to \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < \frac{N+2}{N-2}$, $\epsilon > 0$ is a real parameter and a^+, a^- denote, respectively, the positive and the negative part of a sign-changing function $a : \overline{\Omega} \to \mathbb{R}$. Notice that, in view of these assumptions, the equation in (1.1) is *superlinear indefinite*. The existence of positive solutions, satisfying Dirichlet or Neumann boundary conditions, has been the object of extensive investigation in the last decades, starting with the pioneering papers [2,8] (see also for [21,22] for a wide bibliography on the subject). Boundary blow-up conditions, on the other hand, were taken into account in [31] and [33] in order to describe the limit profile of solutions to parabolic problems like:

$$\begin{cases} u_t - \Delta u = \lambda u + a(x)u^p & \text{in } \Omega \times (0,+\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0 > 0 & \text{in } \Omega \end{cases}$$

when the weight function a is allowed to change sign in a suitable way inside the domain Ω. Roughly speaking, one of the main consequences of the analysis is to outline that the position of λ with respect to the principal eigenvalues of the different nodal regions of the weight function a is fundamental to determine the behavior of the solutions. We address the reader in particular to the very recent monograph [32] for further details and a thorough discussion on the relevance of large solutions in this context.

In [24], which corresponds to $\lambda = 0$ with respect to the discussion carried above, the main result asserts – under some additional technical assumptions – the existence of *two* positive solutions to (1.1) when $\epsilon > 0$ is sufficiently small. Incidentally, we notice that, via a standard rescaling, the same result is true for positive blowing-up solutions of the equation