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The existence of minimum speed of traveling wave solutions to a non-KPP isothermal diffusion system

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Abstract

The reaction–diffusion system $a_t = a_{xx} - ab^n$, $b_t = Db_{xx} + ab^n$, where $n \geq 1$ and $D > 0$, arises from many real-world chemical reactions. Whereas $n = 1$ is the KPP type nonlinearity, which is much studied and very important results obtained in literature not only in one dimensional spatial domains, but also multi-dimensional spaces, but $n > 1$ proves to be much harder. One of the interesting features of the system is the existence of traveling wave solutions. In particular, for the traveling wave solution $a(x,t) = a(x - vt)$, $b(x,t) = b(x - vt)$, where $v > 0$, if we fix $\lim_{x\to-\infty}(a, b) = (0, 1)$ it was proved by many authors with different bounds $v_*(n, D) > 0$ such that a traveling wave solution exists for any $v \geq v_*$ when $n > 1$. For the latest progress, see [7]. That is, the traveling wave problem exhibits the mono-stable phenomenon for traveling wave of scalar equation $u_t = u_{xx} + f(u)$ with $f(0) = f(1) = 0$, $f(u) > 0$ in $(0, 1)$ and, $u = 0$ is unstable and $u = 1$ is stable. A natural and significant question is whether, like the scalar case, there exists a minimum speed. That is, whether there exists a minimum speed $v_{min} > 0$ such that traveling wave solution of speed v exists iff $v \geq v_{min}$? This is an open question, in spite of many works on traveling wave of the system in last thirty years. This is duo to the reason, unlike the KPP case, the minimum speed cannot be obtained through linear analysis at equilibrium points $(a, b) = (0, 1)$ and $(a, b) = (1, 0)$. In this work, we give an affirmative answer to this question.

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1. Introduction

In this paper we study the reaction–diffusion system

$$(I)\begin{cases}\dfrac{\partial a}{\partial t}=\dfrac{\partial^2 a}{\partial x^2}-ab^n,\\ \dfrac{\partial b}{\partial t}=D\dfrac{\partial^2 b}{\partial x^2}+ab^n,\end{cases}$$

where $n > 1$, $D > 0$ and (a, b) are non-negative smooth functions with continuous initial values (a_0, b_0). In isothermal diffusion, a is the density of material consumed, b is the temperature, see [5,4,15]. In addition, it models, after a simple scaling, a simple autocatalytic chemical reaction of the form

$$A + nB \to (n + 1)B \qquad \text{with rate} \quad kab^n,$$

where $k > 0$ is a rate constant, between two chemical species A and B. More importantly, the system arises from many important chemical wave models of excitable media ranging from the idealized Brusselator to real-world clock reactions such as Belousov–Zhabotinsky reaction, the Briggs–Rauscher reaction, the Bray–Liebhafsky reaction and the iodine clock reaction. In that setting, its importance was recognized fairly early, [13,12,22]. Another type of application is that of biological pattern formation of Turing type. In particular, for the purpose of replicating experimental results in early 1990s, two significant models CIMA and Gray–Scott both have the special case of $n = 2$ incorporated in the complete system, see [17,18].

One interesting feature of the system is the existence of traveling waves, which describes the spreading of chemical species B, when locally added to the uniformly distributed A, by consuming chemical species A. This phenomenon was observed in experiment, see [14,22].

By using the simple scaling invariance property of the system and a conservation law, see [7], we can normalize the traveling wave problem, with $a(x, t) = \alpha(x - vt)$, $b(x, t) = \beta(x - vt)$ as follows.

$$\begin{cases}\alpha' + D\beta' = v(1 - \alpha - \beta), \qquad \alpha' \geqslant 0 \qquad \forall z \in \mathbb{R},\\ D\beta'' + v\beta' = -\alpha\beta^n, \qquad \forall z \in \mathbb{R},\\ \lim_{z\to\infty}(\alpha, \beta) = (1, 0),\\ \lim_{z\to-\infty}(\alpha, \beta) = (0, 1),\end{cases} \tag{1.1}$$

where $v > 0$ is the speed of traveling wave. Here $\alpha' \geqslant 0$ is not an additional restriction, because for a traveling wave solution, it is direct from (I) that $\alpha'' + c\alpha' = \alpha\beta^n$ and therefore $a' \geqslant 0$.

There have been many works on the existence of traveling waves of (I) and related models in the last thirty years, [2,7–9,11,16,19–21]. A typical result is there exist two constants $v_1(n, D) > v_2(n, D) > 0$ such that if $v \geq v_1$ there exists a traveling wave with speed v; but if $v < v_2$, there does not exist a traveling wave. The emphasize is on estimating the smallest-speed traveling wave and close the gap between v_1 and v_2. The situation is very different from the case of $n = 1$,