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Nonlocal problems in thin domains

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Abstract

In this paper we consider nonlocal problems in thin domains. First, we deal with a nonlocal Neumann problem, that is, we study the behavior of the solutions to $f(x) = \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))dy$ with $J_\epsilon(z) = J(z_1, \epsilon z_2)$ and $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ a bounded domain. We find that there is a limit problem, that is, we show that $u^\epsilon \to u_0$ as $\epsilon \to 0$ in Ω and this limit function verifies $\int_{\Omega_2} f(x_1, x_2)\, dx_2 = |\Omega_2| \int_{\Omega_1} J(x_1 - y_1, 0)(U_0(y_1) - U_0(x_1))dy_1$, with $U_0(x_1) = \int_{\Omega_2} u_0(x_1, x_2)\, dx_2$. In addition, we deal with a double limit when we add to this model a rescale in the kernel with a parameter that controls the size of the support of J. We show that this double limit exhibits some interesting features.

We also study a nonlocal Dirichlet problem $f(x) = \int_{\mathbb{R}^N} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))\, dy$, $x \in \Omega$, with $u^\epsilon(x) \equiv 0$, $x \in \mathbb{R}^N \setminus \Omega$, and deal with similar issues. In this case the limit as $\epsilon \to 0$ is $u_0 = 0$ and the double limit problem commutes and also gives $v \equiv 0$ at the end.

MSC: 45A05; 45C05; 45M05

Keywords: Thin domains; Nonlocal equations; Neumann problem; Dirichlet problem

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1. Introduction

Our main goal in this paper is to study nonlocal problems with non-singular kernels in thin domains. We deal with Neumann or Dirichlet conditions.

The Neumann problem. We consider

$$f(x) = \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))dy \tag{1.1}$$

where

$$J_\epsilon(z) = J(z_1, \epsilon z_2),$$

$\epsilon > 0$ is a parameter, and $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is a bounded Lipschitz domain. We denote by $x = (x_1, x_2)$ a point in $\Omega_1 \times \Omega_2$, that is, $x_1 \in \Omega_1$, $x_2 \in \Omega_2$.

Here, and along the whole paper, the function J satisfies the following hypotheses

$$J \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \text{ is non-negative with } J(0) > 0,\ J(-x) = J(x) \text{ for every } x \in \mathbb{R}^N, \text{ and}$$
$$\int_{\mathbb{R}^N} J(x)\,dx = 1. \tag{H}$$

On the other hand we only assume that $f \in L^2(\Omega)$.

It is worth mentioning that we are calling (1.1) as a nonlocal thin domain problem due to its equivalence with the equation

$$h_\epsilon(z_1, z_2) = \frac{1}{\epsilon^{N_2}} \int_{\Omega_1 \times \epsilon\,\Omega_2} J(z-w)(v(w) - v(z))\,dw \tag{1.2}$$

with $h_\epsilon(z_1, z_2) = f(z_1, z_2/\epsilon)$. This equivalence between (1.2) and (1.1) is a direct consequence of the change of variable

$$\Omega_1 \times \Omega_2 \ni (x_1, x_2) \mapsto (x_1, \epsilon x_2) \in \Omega_1 \times \epsilon\,\Omega_2.$$

Furthermore, we can see that (1.2), and then (1.1), are nonlocal singular problems since the bounded domain $\Omega_1 \times \epsilon\,\Omega_2$ degenerates to $\Omega_1 \times \{0\}$ when the positive parameter ϵ goes to zero. We also are in agreement with references [8,17,14] using the factor $1/\epsilon^{N_2}$ in (1.2) to preserve the relative size of the open set $\Omega_1 \times \epsilon\,\Omega_2$ for small ϵ (notice that in terms of the Lebesgue measure in $\mathbb{R}^N$ we have that $\epsilon^{-N_2}|\Omega_1 \times \epsilon\,\Omega_2| = |\Omega_1 \times \Omega_2|$). The convenience of dealing with (1.1) is clear since its solutions u^ϵ are defined in the fixed domain $\Omega = \Omega_1 \times \Omega_2$.

Solutions to (1.1) are understood in a weak sense, that is,

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f(x)\varphi(x)\,dx &= \int_{\Omega_1 \times \Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))\,dy\,\varphi(x)\,dx \\ &= -\frac{1}{2} \int_{\Omega_1 \times \Omega_2} \int_{\Omega_1 \times \Omega_2} J_\epsilon(x-y)(u^\epsilon(y) - u^\epsilon(x))(\varphi(y) - \varphi(x))\,dydx, \end{aligned} \tag{1.3}$$

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