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Celestial mechanics solutions that escape ☆

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Abstract

We establish the existence of an open set of initial conditions through which pass solutions without singularities to Newton's gravitational equations in $\mathbb{R}^3$ on a semi-infinite interval in forward time, for which every pair of particles separates like At, $A > 0$, as $t \to \infty$. The solutions are constructable as series with rapid uniform convergence and their asymptotic behavior to any order is prescribed. We show that this family of solutions depends on $6N$ parameters subject to certain constraints.
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1. Introduction

The differential equation governing the position 3-vectors q_i of N point-masses m_i, $i = 1, \dots, N$ moving in $\mathbb{R}^3$ under the influence of their mutual gravitation is

$$\ddot{q}_i = \sum_{j \neq i} \frac{m_j (q_j - q_i)}{\|q_i - q_j\|^3}, \tag{1.1}$$

where the units have been chosen so that Newton's gravitational constant $G = 1$. Each point mass may represent the total mass of an asteroid, comet, planet, star, galaxy, or any distinguished

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aggregate of molecules. Initial conditions accompany (1.1), denoted by $q_i(t_0)$ and $\dot{q}_i(t_0)$ subject to the restrictions that $q_i(t_0) \neq q_j(t_0)$, $i \neq j$.

Our study is concerned with the problem of an expanding universe as defined below.

Definition. We say that *the universe is expanding* if the system of equations (1.1) possess solutions $q_i(t), i = 1, \ldots, N$ that exist on a semi-infinite interval $[t_0, \infty)$ with

$$lim_{t\to\infty}\|q_i(t) - q_j(t)\| = \infty, \; i \neq j.$$

Solutions to an expanding universe beg numerous questions, including: How large is the family of solutions to an expanding universe? Is it possible to associate with them a series representation that is absolutely and uniformly convergent on a semi-infinite interval? How fast is the convergence? Is it possible to obtain approximations as well as asymptotic approximations to any order of accuracy? The goal of this article is to provide some answers to these questions as given in the theorem below. This is preceded by some notation and the context for our existence and asymptotic results.

Notation convention: We will use variables without subscripts, such as δ, to refer to the ordered set of vectors $\{\delta_1, \ldots, \delta_N\}$. Formally, for each i, $\delta_i = \left(\delta_{ix}, \delta_{iy}, \delta_{iz}\right)^{\dagger}$ is a column 3-vector, and

$$\delta = \left(\delta_1^{\dagger}, \delta_2^{\dagger}, \ldots, \delta_N^{\dagger}\right)^{\dagger}$$

is a column $3N$-vector. Note also that we follow Pollard [11] in using $\log t$ to refer to the natural logarithm of t, rather than the usual convention, $\ln t$. We also put $\overrightarrow{0}$ for the zero vector that is either 3-dimensional or $3N$-dimensional and normally denote by I the identity matrix unless specified otherwise.

The main theorem of this article is described below and is followed by comments and literature comparisons.

Theorem 1. *Given any set of masses m_i and constant 3-vectors a_i and c_i, $i = 1, \ldots, N$, satisfying $\|a_j - a_i\| \neq 0$, $i \neq j$, the differential system (1.1) possesses unique vector solutions*

$$q_i = a_i t + b_i \log t + c_i + \delta_i(t) \; i = 1, \ldots, N, \tag{1.2}$$

on a semi-infinite interval $[t_5, \infty)$ where $t_5 \geq 1$, and $q_i \in C^{\infty}[t_5, \infty)$. The $3N$ coefficients b_i are uniquely determined by the $3N$ coefficients a_i as follows

$$b_i = -\sum_{j\neq i} \frac{m_j\left(a_j - a_i\right)}{\left\|a_j - a_i\right\|^3}, \; i = 1, \ldots, N. \tag{1.3}$$

Each vector $\delta_i(t)$ is approximated to any level of accuracy by a sequence $\delta_i[n]$ and computed successively by certain iterations,

$$\delta_i(t) - \delta_i[n](t) = O\left(\frac{K_4^n}{[(n+1)!]^2}\frac{\log t}{t^{n+1}}\right), \tag{1.4}$$