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Journal of Differential Equations

www.elsevier.com/locate/jde

The critical semilinear elliptic equation with isolated boundary singularities ☆

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Received 28 October 2016; revised 25 February 2017

Abstract

We establish quantitative asymptotic behaviors for nonnegative solutions of the critical semilinear equation $-\Delta u = u^{\frac{n+2}{n-2}}$ with isolated boundary singularities, where $n \geq 3$ is the dimension.

MSC: 35J60; 35B40

Keywords: Critical semilinear elliptic equations; Boundary isolated singularity; Asymptotic analysis

1. Introduction

The internal isolated singularity for positive solutions of the semilinear equation $-\Delta u = u^p$ has been very well understood, where Δ is the Laplace operator, $1 < p \leq \frac{n+2}{n-2}$ is a parameter and $n \geq 3$ is the dimension. See Lions [20] for $1 < p < \frac{n}{n-2}$, Gidas–Spruck [12] for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, Aviles [1] for $p = \frac{n}{n-2}$, Caffarelli–Gidas–Spruck [8] for $\frac{n}{n-2} \leq p \leq \frac{n+2}{n-2}$ and Korevaar–Mazzeo–Pacard–Schoen [15] for $p = \frac{n+2}{n-2}$. The Sobolev critical exponent $p = \frac{n+2}{n-2}$ case is of particular interest, because the equation connects to the Yamabe problem and the conformal invariance leads to a richer isolated singularity structure. See also Li [17] and Han–Li–Teixeira [13] for conformally invariant fully nonlinear elliptic equations.

☆ Supported in part by NSFC 11501034, NSFC 11571019, key project of NSFC 11631002 and the Fundamental Research Funds for the Central Universities.

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http://dx.doi.org/10.1016/j.jde.2017.03.034

Please cite this article in press as: J. Xiong, The critical semilinear elliptic equation with isolated boundary singularities, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.03.034

The Dirichlet boundary isolated singularity for the same equation has also been studied in many cases. Asymptotic behaviors of singular solutions have been established by Bidaut–Véron–Vivier [5] for $1 < p < \frac{n+1}{n-1}$ and Bidaut–Véron–Ponce–Véron [3,4] for $\frac{n+1}{n-1} \le p < \frac{n+2}{n-2}$. Existence of singular solutions vanishing on boundaries of bounded domains except finite points has been obtained by del Pino–Musso–Pacard [11] for $p < \frac{n+2}{n-2}$. The exponent $\frac{n+1}{n-1}$ corresponding to $\frac{n}{n-2}$ for the interior singularity was discovered by Brézis–Turner [6]. Under a blow up rate assumption Bidaut–Véron–Ponce–Véron [3,4] obtain refined asymptotic behaviors for the supercritical case $\frac{n+2}{n-2} < p < \frac{n+1}{n-3}$. We refer to [3] and references therein for related results on boundary singularity.

This paper is concerned with the remaining critical case: $p = \frac{n+2}{n-2}$. The conformal invariance again produces additional complexity and the boundary condition makes the asymptotic analysis of [8] and [15] fail. As said in Bidaut–Véron-Ponce–Véron [4], one can show

Proposition 1.1. *Denote* $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. *Let* $u \in C^2(\mathbb{R}^n_+) \cap C(\bar{\mathbb{R}}^n_+ \setminus \{0\})$ *be a nonnegative solution of*

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } \mathbb{R}^n_+, \\ \quad u = 0 & \text{on } \partial\mathbb{R}^n_+ \setminus \{0\}. \end{cases} \tag{1}$$

Suppose 0 *is a non-removable singularity of* u*, then* u *depends only on* $|x'|$ *and* x_n*, and* $\partial_r u(r, x_n) < 0$ *for all* $r = |x'| > 0$.

Note that nothing about the behavior of u at infinity is assumed in Proposition 1.1.

Let u be a solution of (1) and define $U(t, \theta) := |x|^{\frac{n-2}{2}} u(|x| \cdot \theta)$ with $t = -\ln|x|$. Then we have

$$\partial^2_{tt} U + \Delta_{\mathbb{S}^{n-1}} U - \frac{(n-2)^2}{4} U + n(n-2)U^{\frac{n+2}{n-2}} = 0 \quad \text{on } \mathbb{R} \times \mathbb{S}^{n-1}_+, \tag{2}$$

$$U = 0 \quad \text{on } \mathbb{R} \times \partial\mathbb{S}^{n-1}_+, \tag{3}$$

where $\mathbb{S}^{n-1}_+ = \{\theta = (\theta_1, \dots, \theta_n) \in \mathbb{S}^{n-1} : \theta_n > 0\}$. By Proposition 1.1, $U(t, \theta) = U(t, \theta_n)$. In contrast to the internal singularity studied by Caffarelli–Gidas–Spruck [8] and Korevaar–Mazzeo–Pacard–Schoen [15], we lose ODE analysis to classify all solutions of equation (2)–(3). del Pino–Musso–Pacard [11] conjectured that there exists a one-parameter family of periodic solutions of (2)–(3). Bidaut–Véron–Ponce-Véron [3,4] proved that there exists a unique t-independent solution. Existence of t-dependent solutions and a priori estimates are left open.

Let ψ be a C^2 function in $\mathbb{R}^{n-1}$ satisfying

$$\psi(0) = 0, \quad \nabla\psi(0) = 0.$$

Let $Q_R = \{x = (x', x_n) : x_n > \psi(x')\} \cap B_R$ and $\Gamma_R = \{x = (x', x_n) : x_n = \psi(x')\} \cap B_R$, where B_R is the open ball center at 0 with radius R. We consider nonnegative solutions of

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } Q_1, \\ \quad u = 0 & \text{on } \Gamma_1 \setminus \{0\}. \end{cases} \tag{4}$$

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