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On the regularity of the free boundary in the p-Laplacian obstacle problem ☆

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Abstract

We study the regularity of the free boundary in the obstacle for the p-Laplacian, $\min\{-\Delta_p u,\ u-\varphi\}=0$ in $\Omega\subset\mathbb{R}^n$. Here, $\Delta_p u=\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$, and $p\in(1,2)\cup(2,\infty)$.

Near those free boundary points where $\nabla\varphi\neq 0$, the operator Δ_p is uniformly elliptic and smooth, and hence the free boundary is well understood. However, when $\nabla\varphi=0$ then Δ_p is singular or degenerate, and nothing was known about the regularity of the free boundary at those points.

Here we study the regularity of the free boundary where $\nabla\varphi=0$. On the one hand, for every $p\neq 2$ we construct explicit global 2-homogeneous solutions to the p-Laplacian obstacle problem whose free boundaries have a corner at the origin. In particular, we show that the free boundary is in general not C^1 at points where $\nabla\varphi=0$. On the other hand, under the "concavity" assumption $|\nabla\varphi|^{2-p}\Delta_p\varphi<0$, we show the free boundary is countably $(n-1)$-rectifiable and we prove a nondegeneracy property for u at all free boundary points.

MSC: 35R35

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1. Introduction

In this paper we study the obstacle problem

$$\min\{-\Delta_p u,\, u-\varphi\} = 0 \qquad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{1.1}$$

for the p-Laplacian operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \qquad 1 < p < \infty.$$

The problem appears for example when considering minimizers of the constrained p-Dirichlet energy

$$\inf\left\{ \int_\Omega |\nabla v|^p \,:\, v \in W^{1,p}(\Omega), \quad v \geq \varphi \text{ in } \Omega, \quad v = g \text{ on } \partial\Omega \right\},$$

where φ and g are given smooth functions and Ω is a bounded smooth domain.

The regularity of solutions to (1.1) was recently studied by Andersson, Lindgren, and Shahgholian in [1]. Their main result establishes that if $\varphi \in C^{1,1}$ then

$$\sup_{B_r(x_0)} (u-\varphi) \leq Cr^2 \qquad \text{for all} \quad r \in (0,1)$$

at any free boundary point $x_0 \in \partial\{u > \varphi\}$. Thus, solutions u leave the obstacle φ in a $C^{1,1}$ fashion at free boundary points x_0.

Notice that, near any free boundary point $x_0 \in \partial\{u > \varphi\}$ at which $\nabla\varphi(x_0) \neq 0$, the solution u will satisfy $\nabla u \neq 0$ as well and hence the operator $\Delta_p u$ is uniformly elliptic in a neighborhood of x_0. Therefore, by classical results [2,3,8], the solution u is $C^{1,1}$ near x_0, and the structure and regularity of the free boundary is well understood.

Thus, the main challenge in problem (1.1) is to understand the regularity of solutions and free boundaries near those free boundary points $x_0 \in \partial\{u > \varphi\}$ at which $\nabla\varphi(x_0) = 0$. Our first main result is the following.

Theorem 1.1. *Let $p \in (1,2) \cup (2,\infty)$, and let $\varphi(x) = -|x|^2$ in $\mathbb{R}^2$. There exists a 2-homogeneous function $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.1) in all of $\mathbb{R}^2$, and such that the set $\{u > \varphi\}$ is a cone with angle*

$$\theta_0 = 2\pi\left(1 - \sqrt{\frac{p-1}{2p}}\right) \neq \pi.$$

In particular, the free boundary has a corner at the origin.